UNIVERSITÉ DE MONTRÉAL

ADJOINT-BASED ERROR ESTIMATION FOR THE FRONT-TRACKING METHOD

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Ce mémoire intitulé :

ADJOINT-BASED ERROR ESTIMATION FOR THE FRONT-TRACKING METHOD

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RÉSUMÉ

Ce mémoire contient une preuve partielle de l'existence de l'adjoint pour le problème de l'estimation d'erreur de solutions approximatives de lois de conservation. L'existence de l'adjoint permet une estimation de l'erreur par rapport à une fonctionnelle de la solution qui est souvent une quantité d'intérêt pratique. Le résultat se limite au cas d'une loi de conservation scalaire, en une dimension spatiale et ne contenant que des ondes de chocs. On se limite à estimer l'erreur dans les approximations de front-tracking.

L'approche est unique puis qu'en se limitant aux approximations de front-tracking de Risebro (Risebro, 1993) et Bressan (Bressan, 1992), elle offre la possibilité de s'étendre aux systèmes de lois de conservation nonlinéaires. En ce moment, le seul résultat d'existence de l'adjoint viennent de Tadmor (Tadmor, 1991) où il utilise des techniques qui se limitent aux lois scalaires en 1-D. L'adjoint est obtenu grâce à une nouvelle construction duale à celle de front-tracking. On ne démontre pas la convergence de cet adjoint vers l'adjoint de la solution d'entropie.

ABSTRACT

We show how to measure the error in a functional of the front-tracking approximations by using the adjoint method. This work is restricted to nonlinear scalar conservation laws containing only shock waves. We construct a piecewise linear approximation to the adjoint of the error equation for the piecewise constant front-tracking approximations. We show that under certain conditions, this can be used to bound the error. The construction of the adjoint is new and based on a scheme that is dual to the front-tracking scheme.

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u:	Entropy solution of Cauchy problem	page 15
L:	Lipschitz constant of flux f	page 18
TV:	Total variation	page 18
L^1:	$\lVert u\rVert_{L^1} = \int_\Omega \lvert u(x)\rvert dx < \infty$	page 18
L^∞:	$\lVert u\rVert_{L^\infty} = \max \lvert u\rvert < \infty$	page 18
$W^{1,\infty}$:	$\lVert u\rVert_{W^{1,\infty}} = \lVert u\rVert_{L^1} + \lVert \frac{\partial u}{\partial x}\rVert_{L^\infty}$	page 19
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ψ:	Test function	page 36
ϕ:	Test function	page 37
$R(u^\epsilon, \phi)$:	Weak form of the residual of the approximation	page 37
ζ:	Adjoint with initial data ζ_0 at time T	page 38
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$\xi^{\eta,h}$:	A piecewise linear approximation of ζ	page 39
$F(u^\epsilon, u^\eta)$:	Flux of the adjoint problem	page 42
N:	Number of interpolation nodes in $\xi^{\eta,h}(x,T)$	page 50
$M(t)$:	Total number of shocks in u^ϵ and in u^η at time t	page 50
$I(0)$:	Number of shock crossings	page 50
$P_i(t)$:	The characteristic through the i-th shock crossing	page 50

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CONDENSÉ EN FRANÇAIS

Introduction

La simulation précise et efficace de l'écoulement de fluides compressibles est fondamentale en science. En fait, les lois de conservation non linéaires qui sont utilisées pour modéliser ces fluides permettent aussi de modéliser d'autres phénomènes comme certains types de déformations élastiques et le déclenchement d'explosions nucléaires. Dans tous ces cas, on recherche l'estimation précise d'une fonctionelle $J(u)$ de la solution exacte u. Par exemple, on pourrait rechercher la portée d'une aile d'avion, la pression à la sortie d'un moteur d'avion ou le temps avant la détonation d'une explosion. La méthode adjointe permet l'estimation de l'erreur locale $J(u) - J(u_h)$ associée à l'approximation u_h. Les autres approches pour l'estimation de l'erreur ne permettent que l'estimation de l'erreur globale $||u - u_h||$ par rapport à une certaine norme. L'information obtenue par ces autres approches ne guarantit pas une réduction de l'erreur $J(u) - J(u_h)$. De plus, si on réduit de manière sélective la taille h des éléments qui contribuent à l'erreur globale $||u - u_h||$, on risque de réduire un trop grand nombre d'éléments et par conséquent d'adapter le maillage d'une manière sous-optimale.

Malheureusement, dans le cas des lois de conservation non linéaires, l'analyse rigoureuse de la méthode adjointe est très difficile puisqu'il n'existe aucune théorie de la stabilité à laquelle se rattacher. Dans ce mémoire, on fait un premier pas vers le développement d'une technique rigoureuse pour l'analyse de la méthode adjointe pour les lois de conservation non linéaires. On étudie l'application de la méthode adjointe à l'estimation des erreurs numériques pour l'estimation des erreurs dans les approximations dites de front-tracking. On construit une nouvelle méthode numérique pour estimer l'adjoint de ces approximations et donc l'erreur par rapport à une fonctionelle de la solution. On montre que sous certaines hypothèses, ces approximations convergeront vers la solution du problème adjoint associée à l'estimation de l'erreur $J(u) - J(u_h)$. L'approche choisie ici offre la possibilité de traiter des systèmes de lois de conservation.

Les lois de conservation

Une solution lisse d'une loi de conservation est une solution du problème de Cauchy

$$\begin{aligned} u_t + f(u)_x &= 0, \\ u(\cdot, 0) &= u_0(\cdot), \end{aligned} \tag{1}$$

où f est le flux et u_0 sont les conditions initiales. Nous nous restreindrons au cas où u et f sont des fonctions à valeurs réelles et x appartient à un espace de dimension un.
Si une courbe $x(t)$ satisfait l'équation

$$\begin{aligned} x'(t) &= f'(u(x,t)), \\ x(0) &= x_0, \end{aligned}$$

alors la variation de u le long de la courbe sera $d/dt(u(x(t), t)) = u_x dx/dt + u_t = 0$, c-à-d que u est constant le long de ces courbes. On appelle de telles courbes des *caractéristiques*.
Malheureusement, si f est non linéaire, avec par exemple $f''(u) > 0$ et $du_0/dx < 0$, alors les caractéristiques convergeront et la solution deviendra indéterminée. Sans rentrer dans les détails qui sont contenus dans la Section 1.2, il est en effet possible de montrer que si f est non linéaire, alors même avec des conditions initiales lisses, la solution peut développer des discontinuités (Bressan, 2001).

Pour une équation différentielle, il est habituellement impossible de donner un sens à une solution quand cette solution devient discontinue. Les lois de conservation se distinguent parmi les équations différentielles justement par le fait qu'il existe un sens claire à la notion de solution non-différentiable.

Définition *Étant donnée une condition initiale* u_0*, une fonction mesurable* $u = u(x,t)$ *est dite une solution de la loi de conservation*

$$u_t + f(u)_x = 0,$$

au sens faible si $\forall \phi \in W^{1,\infty}(\mathbb{R} \times [0,T])$ *avec support compact*

$$\int_{\mathbb{R}\times[0,T]} u \cdot \phi_t + f(u) \cdot \phi_x \, dxdt + \int_{\mathbb{R}} u_0(x) \cdot \phi(x,0)\, dx = \int_{\mathbb{R}} u(x,T) \cdot \phi(x,T)\, dx.$$

Définition *Étant donnée une solution de la loi de conservation au sens faible, la fonction* u *est une solution d'entropie si pour toute paire de fonctions* ψ *et* μ*, avec* μ *convexe et satisfaisant* $\psi'(u) = \mu'(u) \cdot f'(u)$*, on a pour tout* $\phi \in W^{1,\infty}(\mathbb{R} \times (0,T))$ *à support compact,*

$$\int_{\mathbb{R}\times(0,T)} \mu(u) \cdot \phi_t + \psi(u) \cdot \phi_x \, dxdt \geq 0.$$

Dans le cas des lois de conservation scalaires avec flux convexe et dans la classe des solutions appartenant à $L^\infty_{\mathrm{loc}}\big([0,T], \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^n)\big)$, ensemble ces deux conditions sont essentiellement équivalentes aux conditions de Rankine-Hugoniot et d'entropie de Lax (voir Section 1.2) (LeFloch, 2002). On rappelle la définition de la variation totale d'une fonction $\phi : [a,b] \to \mathbb{R}$(voir Section 1.2) :

$$TV\{\phi\} := \inf_{a=x_0<x_1<\ldots<x_{n-1}<x_n=b} \sum_{i=1}^{n} \left|f(x_i) - f(x_{i-1})\right|.$$

Théorème *Soit f localement Lipschitz continue et $u_0 \in L^1$ à variation bornée. Alors il existe une unique solution faible d'entropie $u = u(x,t)$, satisfaisant*

$$TV\{u(\cdot,t)\} \leqslant TV\{u_0\}, \ \|u(\cdot,t)\|_{L^\infty} \leqslant \|u_0\|_{L^\infty} \qquad \textit{pour toutt } \geqslant 0.$$

Il existe des constantes M et L telles que si v est une solution d'entropie pour d'autres conditions initiales, alors

$$\begin{aligned} &|u(x,t)| \leqslant M, \ |v(x,t)| \leqslant M && \textit{pour tout } t, x, \\ &|f(w) - f(w')| \leqslant L|w - w'| && \textit{pour tout } w, w' \in [-M, M]. \end{aligned}$$

De plus, pour tout $R > 0$ et $\tau \geqslant \tau_0 \geqslant 0$, nous avons

$$\int_{|x|\leqslant R} |u(x,\tau) - v(x,\tau)|dx \leqslant \int_{|x|\leqslant R+L(\tau-\tau_0)} |u(x,\tau_0) - v(x,\tau_0)|dx. \qquad (2)$$

Dans la Section 2.1, on montre le fait déjà bien connu que les approximations de front-tracking convergent vers la solution d'entropie.

La méthode de front-tracking

La méthode de front-tracking est inspirée du schéma de Glimm (Glimm, 1965) allié à la méthode de trace des ondes de Liu (Liu, 1977). La méthode est le résultat de travaux de

Dafermos, DiPerna, Risebro et Bressan. Elle produit une approximation simple qui est constante par morceaux par rapport à la variable spatiale x.

Nous décrivons brièvement cette construction. Soit $\epsilon > 0$ et une approximation constante par morceaux u_0^ϵ des conditions initiales u_0. Supposons que cette approximation qui possède un nombre fini de discontinuités et satisfait

$$\begin{aligned} \|u_0 - u_0^\epsilon\|_{L^1} &< \epsilon, \\ TV(u_0^\epsilon) &\leqslant TV(u_0). \end{aligned}$$

Pour définir $u^\epsilon(\cdot, t)$ constante par morceaux pour tout $t \geqslant 0$, il suffit de poser $u^\epsilon(\cdot, 0) = u_0^\epsilon$ et définir la vitesse de propagation de chaque discontinuité. Bien sûr, on doit aussi décrire comment deux discontinuités vont interagir quand elles se croiseront.

Supposez qu'il y ait une discontinuité α situé à $x_\alpha(t)$ dans l'approximation de front-tracking. Si les états à gauche et à droite de la discontinuité sont

$$\begin{aligned} u_\alpha^-(t) &= \lim_{x \to x_\alpha^-(t)} u^\epsilon(x, t), \\ u_\alpha^+(t) &= \lim_{x \to x_\alpha^+(t)} u^\epsilon(x, t), \end{aligned}$$

que $u_\alpha^-(t) > u_\alpha^+(t)$ et que f est convexe, alors une telle discontinuité α s'appelle une *onde de choc.* On suppose que la vitesse de propagation satisfait

$$\left|\dot{x}_\alpha(t) - S(u_\alpha^-(t), u_\alpha^+(t))\right| < \epsilon,$$

où S est la vitesse de Rankine-Hugoniot (1.33). Quand $x_\alpha(t) = S(u_\alpha^-(t), u_\alpha^+(t))$ alors il n'y aura pas d'erreur mais pour les besoins de la cause (c-à-d pour créer des erreurs), supposons que la vitesse de propagation des ondes dans l'approximation ne soit pas exacte. Si $0 < u_\alpha^+(t) - u_\alpha^-(t) \leqslant \epsilon$, alors la discontinutié α s'appelle une *onde de détente*. Si $0 < \epsilon < u_\alpha^+(t) - u_\alpha^-(t)$, alors on subdivisera la discontinuité en k nouvelles discontinuités séparées par des états intermédiaires

$$u_\alpha^-(t) = u^{(0)} < u^{(1)} < u^{(2)} < \cdots < u^{(k)} = u_\alpha^+(t)$$

afin que $u^{(i)} - u^{(i-1)} < \epsilon$ et que chaque discontinuité à $y^{(i)}(t)$, avec états à droite et à gauche $u^{(i-1)}$, $u^{(i)}$, satisfasse la condition

$$\left|\dot{y}_\alpha^{(i)}(t) - S(u^{(i-1)}, u^{(i)})\right| < \epsilon.$$

On suppose aussi que les ondes de détentes voisines ne se croisent pas.
Si deux discontinuités se rencontrent au temps t^*, alors le résultat est encore une discontinuité et on utilise l'algorithme précédent pour décrire la propagation de l'onde résultante. Bien que l'approximation des ondes de détentes dans u^ϵ par des discontinuités ne soit pas physiquement acceptables, la taille de ces discontinuités est bornée par ϵ. De plus, pour les grandes ondes de détentes continues, on peut approximer de manière raisonnable le profil continu si on se sert d'un grand nombre de petite discontinuités. Pour simplifier notre travail, nous supposerons que les discontinuités ont des vitesses constantes par rapport au temps.

La méthode adjointe

On cherche à estimer l'erreur dans l'évaluation de $J(u)$ quand u est remplacé par une approximation u^ϵ. Pour simplifier notre approche, les fonctionelles considérées seront de la forme

$$J(u) = \int_{\mathbb{R}} u(x,T) \cdot \zeta_0(x)\, dx,$$

mais on pourrait traiter des fonctionnelles qui dépendent autant de x que de t. Quand ζ_0 est une fonction positive de masse 1, disons $\zeta_0(x) \approx C \cdot e^{-(x-x_0)^2}$, alors la fonctionnelle J mesure la valeur moyenne de u dans un petit voisinage de x_0. De même, quand $\zeta_0(x) \approx C \cdot (x - x_0) \cdot e^{-(x-x_0)^2}$ alors on peut mesurer la dérivée de u en x_0. Plus généralement, J peut servir à mesurer le flux sortant dans une région du domaine.

Pour toute fonction test ϕ, posons

$$\begin{aligned} R(u^\epsilon, \phi) \doteq & -\int_{\mathbb{R}\times[0,T]} u^\epsilon \cdot \phi_t + f(u^\epsilon) \cdot \phi_x \, dxdt \\ & - \int_{\mathbb{R}} u^\epsilon(x,0) \cdot \phi(x,0)\, dx + \int_{\mathbb{R}} u^\epsilon(x,T) \cdot \phi(x,T)\, dx. \end{aligned} \tag{3}$$

Clairement, $R(u, \phi) = 0$ si u est une solution faible de la loi de conservation.

Un calcul simple décrit dans la section 2.2 montre que s'il existe une fonction lisse ζ qui satisfait $\zeta(\cdot, T) = \zeta_0(\cdot)$, alors l'erreur s'écrira

$$\begin{aligned} J(u^\epsilon) - J(u) = & \int_{\mathbb{R}\times[0,T]} (u^\epsilon - u) \cdot \zeta_t + \big(f(u^\epsilon) - f(u)\big) \cdot \zeta_x \, dxdt \\ & + \int_{\mathbb{R}} \big(u^\epsilon(x,0) - u(x,0)\big) \cdot \zeta(x,0)\, dx + R(u^\epsilon, \zeta), \end{aligned}$$

où $R(\cdot, \cdot)$ est la forme faible du résidu définie plus tard par la formule (2.11). Si on suppose que ζ satisfait de plus l'équation différentielle

$$\begin{aligned}\zeta_t + F(u, u^\epsilon)\zeta_x = 0,\\ \zeta(\cdot, T) = \zeta_0(\cdot),\end{aligned}$$

où $F(u, u^\epsilon) = (f(u) - f(u^\epsilon))/(u - u^\epsilon)$, et que l'on remplace u (qui n'est pas calculable) par u^η, alors

$$\begin{aligned}J(u^\epsilon) - J(u) = \int_{\mathbb{R}} \big(u^\epsilon(x,0) - u^\eta(x,0)\big)\zeta^\eta(x,0)\,dx + R(u^\epsilon, \zeta^\eta)\\ + \mathcal{O}(\eta) + \mathcal{O}(\epsilon)\cdot\mathcal{O}(\eta) + R(u^\epsilon, \mathcal{O}(\eta)).\end{aligned}$$

Ci-dessus, ζ^η est la solution de l'équation adjointe avec u remplacé par u^η. Si $\eta = \mathcal{O}(\epsilon^2)$, alors les derniers termes seront négligeables et l'erreur sera formée de l'erreur d'approximation initiale et de $R(u^\epsilon, \zeta^\eta)$.

Les résultats

Théorème *Supposons que u^ϵ et u^η soient des approximations de front-tracking pour des conditions initiales décroissantes. Supposons que $\{\xi_0^h(\cdot)\}_h$ soit une suite d'approximations linéaires par morceaux qui converge vers $\zeta_0(\cdot)$ dans $W^{1,\infty}(\mathbb{R})$ lorsque $h \to 0$. Pour chaque valeur positive de h, ϵ et η, il existe une fonction linéaire par morceaux $\xi^{\eta,h} \in W^{1,\infty}(\mathbb{R}\times[0,T])$ de l'équation*

$$\begin{aligned}\xi_t^{\eta,h} + F(u^\epsilon, u^\eta)\xi_x^{\eta,h} &= 0\\ \xi^{\eta,h}(\cdot, T) &= \xi_0^h(\cdot).\end{aligned} \tag{4}$$

Ce résultat montre qu'il existe une façon de construire un adjoint qui est linéaire par morceaux, donc dans $W^{1,\infty}(\mathbb{R} \times [0,T])$. La construction est semblable à celle utilisée pour construire l'approximation initiale u^ϵ. La preuve est basée sur le fait que si u^ϵ et u^η sont constantes par morceaux, alors $F(u^\epsilon, u^\eta)$ le sera aussi. Donc, les caractéristiques de l'équation adjointe seront parallèles. Si les données initiales $\xi^{\eta,h}$ sont linéaires par morceaux alors le transport de $\xi^{\eta,h}$ le long des caractéristiques préservera cette propriété. La difficulté de la preuve est de tenir compte du croisement des caractéristiques de l'équation adjointe avec les discontinuités de u^ϵ et u^η.

Ce résultat ne démontre pas entièrement l'existence d'une solution à l'équation adjointe mentionnée dans la section précédente puisque la vitesse de transport $F(u^\epsilon, u^\eta)$ est constante par morceaux mais le vrai flux $F(u, u^\epsilon)$ ne l'est pas. Ce deuxième résultat découlerait si le théorème suivant était vrai.

Conjecture *Il existe des constantes C_1 et C_2, qui ne dépendent que de f, u_0 et ϵ, telles que pour tout η_1, η_2, h_1, h_2,*

$$\begin{aligned} \|\xi^{\eta_1,h_1} - \xi^{\eta_2,h_2}\|_{W^{1,\infty}(\mathbb{R}\times[0,T])} \leq \quad & C_1\|\xi_0^{h_1} - \xi_0^{h_2}\|_{W^{1,\infty}(\mathbb{R})} \\ & + C_2\|u^{\eta_1} - u^{\eta_2}\|_{\mathrm{Lip}([0,T],L^1(\mathbb{R}))}. \end{aligned}$$

Nous nous limiterons à montrer que cette inégalité impliquerait l'existence d'une solution à l'équation adjointe. Cette borne de stabilité permet de montrer l'existence d'une solution générale de l'équation adjointe car elle permet de montrer que la suite des approximations $\xi^{\eta,h}$ est Cauchy dans l'espace de Banach $W^{1,\infty}(\mathbb{R} \times [0,T])$.

Conclusions

Nous avons montré l'existence d'une solution de l'équation adjointe dans le cas où les données initiales u_0 sont décroissantes et constantes par morceaux et ζ_0 est linéaire par morceaux. Nous avons aussi montrer que si l'on pouvait démontrer la stabilité de ces approximations alors on obtiendrait un résultat d'existence de l'adjoint pour toute condition initiale u_0 décroissante.

On propose une nouvelle manière de construire l'adjoint qui pourrait facilement se faire en parallèle avec la méthode de front-tracking. L'approche semble donc aussi viable numériquement. En revanche, les résultats analytiques doivent être étendus au cas des ondes de détentes avant d'entreprendre une analyse plus numérique de cette approche.

INTRODUCTION

This thesis contains a partial proof of the existence of the adjoint for the error estimation problem for nonlinear conservation laws. The result is limited to scalar conservation laws in 1-D containing only shock waves. The existence is already guaranteed by a result of Tadmor (Tadmor, 1991), but is based on techniques that are limited to scalar problems in 1-D. The result presented here is unique in that, because it is based on the front-tracking approximations of Bressan (Bressan, 1992) and Risebro (Risebro, 1993), it may be extended to systems of nonlinear conservation laws.

The first chapter contains an introduction to a posteriori error estimation and nonlinear conservation laws. The Section 1.3 contains an overview of the literature on the topic. The second chapter begins with a complete description of front-tracking approximations. In particular, it contains a proof of convergence of these approximations to the entropy solution, at least in the case of scalar conservation laws in 1-D. The second chapter ends with a definition of the adjoint problem for linear functionals of solutions to conservation laws. The third chapter demonstrates the existence of the adjoint of the front-tracking approximation u^ϵ when the entropy solution is replaced by a fine front-tracking approximation u^η. The proof, detailed in Section 3.2, describes a scheme dual to front-tracking in order to construct the adjoint. If a certain stability bound holds, then taking the limit as $\eta \longrightarrow 0$, we obtain an adjoint in $W^{1,\infty}$. We don't provide a proof of this stability bound.

CHAPTER 1

ERROR ESTIMATION OF NUMERICAL SOLUTIONS OF CONSERVATION LAWS

1.1 Error Estimation

An important scientific question is to measure the error in the numerical solution of quantitative problems. In this section, we survey some basic methods for error estimation. We emphasize the difference between a priori & posteriori error estimation and the resulting forms of the error estimate.

1.1.1 Basic knowledge on error estimation

Following Bangerth & Rannacher (Bangerth and Rannacher, 2003), to find the solutions $u \in \mathbb{R}^n$ of the linear equation $Au = b$, where $A \in M^{n \times n}(\mathbb{R}), b \in \mathbb{R}^n$, we can use approximations $A_h \to A$, $b_h \to b$ as $h \to 0$ to build a numerical system: $A_h u_h = b_h$. Here the main quantities in error estimation are:

$$\text{approximation error } e := u - u_h,$$

$$\text{truncation error } \tau := A_h u - b_h,$$

$$\text{residual } \rho := b - Au_h.$$

Our goal here is to estimate the approximation error $e = u - u_h$. There are two main approaches.

i) a priori error analysis:

$$A_h e = A_h u - A_h u_h = A_h u - b_h = \tau$$

$$\Rightarrow$$

$$\|e\| \leqslant \|A_h^{-1}\| \cdot \|\tau\| \tag{1.1}$$

ii) a posteriori error analysis:

$$Ae = Au - Au_h = b - Au_h = \rho$$

$$\Rightarrow$$

$$\|e\| \leqslant \|A^{-1}\| \cdot \|\rho\|. \tag{1.2}$$

Comparing both analysis methods, we can see that a priori error analysis depends on the "discrete" operator A_h, which is not easy to control in practice, while a posteriori error analysis depends on the "continuous" operator A.

We now compare a priori & a posteriori error analysis with an example. Consider the differential equation with initial value:

$$u'(t) = f\big(t, u(t)\big), \tag{1.3}$$

$$u(t_0) = u_0,$$

where $t_0 \leqslant t \leqslant T$.

We consider here one-step integration method for the equation (1.3). Suppose

$$t_0 < t_1 < t_2 < \cdots < t_N = T,$$

the subintervals $I_n = [t_{n-1}, t_n]$, and $h = t_n - t_{n-1} = constant$. Let $u(x)$ be the exact solution and

$$u_n \approx u(t_n), \qquad n = 0, \cdots, N$$

be the approximate values obtained by the scheme

$$u_{n+1} = u_n + hF(t_n, u_n, h), \tag{1.4}$$

where F is the numerical approximation of the exact flux f. Then from the local truncation error's definition, we have

$$\tau_{n-1}(u) = \frac{1}{h}\big[u(t_n) - [u(t_{n-1}) + hF\big(t_{n-1}, u(t_{n-1}), h\big)]\big]$$

so, we get

$$u(t_n) = u(t_{n-1}) + hF\big(t_{n-1}, u(t_{n-1}), h\big) + h\tau_{n-1}(u) \tag{1.5}$$

On the other hand, by combining the scheme

$$u_n = u_{n-1} + hF(t_{n-1}, u_{n-1}, h) \tag{1.6}$$

the identities (1.5) - (1.6), and then observing that $e_n = u(t_n) - u_n$, we deduce that

$$e_n = e_{n-1} + h[F(t_{n-1}, u(t_{n-1}), h) - F(t_{n-1}, u_{n-1}, h)] + h\tau_{n-1}(u)$$

Let $L = \sup_{(t,w,h)} \left|\frac{\partial F}{\partial u}(t, w, h)\right|$, where $\xi \in [u(t_{n-1}), u_{n-1}]$ and since τ is very small, so we think $\tau(u) \approx \tau_{n-1}(u) \approx \tau_{n-2}(u) \cdots$, after that, using induction, we can get

$$
\begin{aligned}
|e_n| &\leq (1 + hL)|e_{n-1}| + h|\tau_{n-1}(u)| \\
|e_n| &\leqslant (1 + hL)\big[(1 + hL)|e_{n-2}| + h|\tau_{n-2}(u)|\big] + h|\tau_{n-1}(u)| \\
|e_n| &\leqslant (1 + hL)^2|e_{n-2}| + h\big[(1 + hL) + 1\big]|\tau(u)| \\
&\vdots \\
|e_n| &\leqslant (1 + hL)^n|e_0| + h\Big[(1 + hL)^{n-1} + (1 + hL)^{n-2} + \cdots + (1 + hL) + 1\Big]|\tau(u)| \\
&\leqslant h \cdot \left\{\frac{(1 + hL)^n - 1}{(1 + hL) - 1}\right\} \cdot |\tau(u)| = \frac{1}{L}\Big\{(1 + hL)^n - 1\Big\} \cdot |\tau(u)| \\
&\leqslant \frac{1}{L}\Big\{e^{nhL} - 1\Big\} \cdot |\tau(u)| = \frac{1}{L}\Big\{e^{LT} - 1\Big\} \cdot |\tau(u)|. \qquad (1.7)
\end{aligned}
$$

So, we get

$$|e_n| \leq \frac{1}{L}\Big[e^{LT} - 1\Big] \cdot |\tau(u)|. \qquad (1.8)$$

Here, we can see that the estimation error depends on the Lipschitz constant L of the "discrete" flux F and the truncation error τ. Comparing the error estimate with (1.1), this is typical of a priori error analysis.

For the differential equation (1.3), we can also build an error estimation based on a posteriori error analysis.

Let $I_n = [x_n, x_{n-1})$ and define

$$S_n^\infty = \{\varphi : [t_0, T] \to \mathbb{R}, \qquad \varphi|_{I_n} \in C^\infty(I_n)\}$$
$$S_n^0 = \{\varphi : [t_0, T] \to \mathbb{R}, \qquad \varphi|_{I_n} \in C^0(I_n)\}.$$

Define the jump at t_n to be

$$[\varphi]_n = \varphi(t_n^+) - \varphi(t_n^-),$$

where $\varphi(t_n^-) = \lim_{t\to t_n^-} \varphi(t)$ and $\varphi(t_n^+) = \lim_{t\to t_n^+} \varphi(t)$. Withthese definitions, in Bangerth and Rannacher (Bangerth and Rannacher, 2003) they show that the exact solution satisfies the weak formulation

$$\sum_{n=1}^{N} \Big\{ \int_{I_n} (u' - f(u)) \cdot \varphi(t)\ dt + [u]_{n-1} \cdot \varphi(t_{n-1}^+) \Big\} = 0, \qquad \forall \varphi \in S_n^0. \tag{1.9}$$

We define an approximation $u_h \in S_n^0$ of the exact solution u as the unique solution of

$$\sum_{n=1}^{N} \Big\{ \int_{I_n} (u_h' - f(u_h)) \cdot \varphi(t)\ dt + [u_h]_{n-1} \cdot \varphi(t_{n-1}^+) \Big\} = 0, \quad \forall \varphi \in S_n^0, \tag{1.10}$$

where $u(t_n) = u_h|_{I_n}$. Using (1.9) - (1.10), we remark that the error $e = u - u_h$ satisfies

$$\sum_{n=1}^{N} \Big\{ \int_{I_n} (e' - f(u) + f(u_h)) \cdot \varphi(t)\ dt + [e]_{n-1} \cdot \varphi(t_{n-1}^+) \Big\} = 0, \quad \forall \varphi \in S_n^0. \tag{1.11}$$

Remark, if $g(s) = f(u_h + se)$, then

$$f(u) - f(u_h) = \int_0^1 g'(s) ds = \int_0^1 f'(u_h + se) \cdot e(t)\, ds := \ e(t) \cdot B(t).$$

So, the (1.11) becomes

$$\sum_{n=1}^{N}\Big\{\int_{I_n}\big(e'-eB(t)\big)\cdot\varphi(t)\ dt+[e]_{n-1}\cdot\varphi(t_{n-1}^{+})\Big\}=0,\quad\forall\varphi\in S_n^0. \tag{1.12}$$

Here, we introduce an adjoint z, which is a solution of the following equation with initial value:

$$\sum_{n=1}^{N}\Big\{\int_{I_n}\big(-z'-Bz\big)\cdot\varphi(t)\ dt+[z]_n\cdot\varphi(t_n^{-})\Big\}=0,\quad\forall\varphi\in S_n^\infty \tag{1.13}$$

$$z(T)=\frac{e(T)}{|e(T)|}.$$

If z_h is any approximation in S_n^0 to z, then Bangerth & Rannacher (Bangerth and Rannacher, 2003) show that by taking $\varphi=e$, (1.13) becomes

$$|e(T)|=\sum_{n=1}^{N}\int_{I_N}\big(-u_h'+f(u_h)\big)(z-z_h)\,dt+\sum_{n=1}^{N}-[u_h]_{n-1}(z-z_h)(t_{n-1}^{+}). \tag{1.14}$$

Suppose we define z_h such that

$$z_h|_{I_n}=\frac{1}{h_n}\int_{I_n}z(t)\,dt,$$

then the first item in (1.14) vanishes. By using the Taylor series, we have

$$(z-z_h)(t_{n-1}^{+})=-\frac{1}{2}h_n z'(t_{n-\frac{1}{2}})+\mathcal{O}(h_n^2),$$

then (1.14) becomes

$$
\begin{aligned}
|e(T)| =& \sum_{n=1}^{N} \int_{I_n} \big(- u_h' + f(u_h) \big)(z - z_h)\, dt + \sum_{n=1}^{N} -[u_h]_{n-1}(z - z_h)(t_{n-1}^{+}) \\
\leqslant& \sum_{n=1}^{N} h_n \cdot \left| \frac{1}{h_n} \big[u_h\big]_{n-1} \right| \cdot \frac{1}{2} \cdot h_n \cdot \sup_{I_n} |z'(t)| \\
\leqslant& \frac{1}{2} \cdot \max_{n} \{h_n \rho_n\} \cdot \sum_{n=1}^{N} \int_{I_n} \big|z'(t)\big|\, dt \\
=& \frac{1}{2} \cdot \max_{n} \{h_n \rho_n\} \cdot \int_{0}^{T} \big|z'(t)\big|\, dt.
\end{aligned} \tag{1.15}
$$

Here, we can see the error depends on the value of z, which is a solution of a "continuous" problem (1.13) and the residual ρ. Comparing the error estimate with (1.2), this is typical of a posteriori error analysis.

1.1.2 Error estimation with a linear functional

In many situations of engineering interest, the goal is to compute the functional $J(u)$ of a solution u as efficiently as possible. For example, consider the Euler equations which are used in practice to model gas dynamics. Let $\rho = \rho(x, t)$ be the density, v be the velocity, ρv be the momentum, E be the energy, and p be the pressure; then the equations are

$$
\frac{\partial}{\partial t} \begin{bmatrix} \rho \\ \rho v \\ E \end{bmatrix} + \frac{\partial}{\partial x} \begin{bmatrix} \rho v \\ \rho v^2 + p \\ v(E + p) \end{bmatrix} = 0. \tag{1.16}
$$

The mean surface pressure of a body S is the functional of ρ, v and E:

$$J(\rho, v, E) = \int_S p\, d\vec{n}.$$

It is often possible and efficient to compute an accurate value of $J(\rho, v, E)$ without obtaining an accurate approximation ρ_h, v_h and p_h. This because $J(\rho, v, E)$ depends on a subset of the approximated data ρ, v, p and is also an averaged quantity.

We now use a duality argument well-known from the error analysis of Galerkin Finite Element Methods to show how to estimate the error with respect to a functional of the solution. Details can be found in Bangerth & Rannacher (Bangerth and Rannacher, 2003). Consider a linear functional $J(\cdot) = (\cdot, j)$, where j is some vector in $\mathbb{R}^n$. We want to estimate the error with respect to J, that is, estimate $J(u) - J(u_h) = J(e)$. Let $z \in \mathbb{R}^n$ be the solution of the linear adjoint problem: $A^* z = j$ and define $\rho = Ae$, then we have

$$J(e) = (e, j) = (e, A^* z) = (Ae, z) = (\rho, z).$$

So, we get

$$\left| J(u) - J(u_h) \right| = |J(e)| \leqslant \sum_{k=1}^{n} |\rho_k| \cdot |z_k|,$$

where ρ_k and z_k are the components of the vectors ρ and z.s

For nonlinear problems: $A(u)^*z = j$, by using the Jacobian matrix $A'(u)$, we have

$$\begin{aligned}\big(A(u) - A(u_h), \phi\big) &= \int_0^1 \big(A'(u_h + se)e, \phi\big)\, ds \\ &= (Be, \phi), \qquad\qquad \text{for any } \phi \in \mathbb{R}^n\end{aligned}$$

where $B := B(u, u_h) = \int_0^1 A'(u_h + se)\, ds$. Then let the adjoint z be the solution of $B^*z = j$, so, we get

$$J(e) = (e, j) = (e, B^*z) = (Be, z) = \big(A(u) - A(u_h), z\big) = (\rho, z).$$

So, we can also get a similar inequality

$$\big|J(u) - J(u_h)\big| = |J(e)| \leqslant \sum_{k=1}^{n} |\rho_k| \cdot |z_k|.$$

But in this case, we cannot control the adjoint z which depends on the unknown exact solution u from the definition of B. Thus, we must use an approximate matrix $\widetilde{B}$ to adjust the adjoint as $\widetilde{z}$

$$B(u, u_h) \approx \widetilde{B} := B(u_h, u_h) = \int_0^1 A'(u_h)\, ds = A'(u_h). \tag{1.17}$$

The nonlinear problem is then approximated by a linear problem :

$$A'(u_h)^* \cdot \widetilde{z}_h = j_h,$$

and

$$\left|(\rho, \widetilde{z}_h)\right| \leqslant \sum_{k=1}^{n} \left|\widetilde{z}_{h,k}\right| \cdot \left|\rho_k\right|.$$

Now, let $\widetilde{z}$ be the solution of

$$\widetilde{B}^*\widetilde{z} = A(u_h)^*\widetilde{z} = j, \tag{1.18}$$

then, we have

$$\begin{aligned} |J(e)| = \left|(e, j)\right| = \left|(e, \widetilde{B}^*\widetilde{z})\right| &= \left|(\widetilde{B}e, \widetilde{z})\right| \\ &\leqslant \left|\left((\widetilde{B} - B)e, \widetilde{z}\right)\right| + \left|(Be, \widetilde{z})\right| \\ &\leqslant \left|\left((\widetilde{B} - B)e, \widetilde{z}\right)\right| + \left|(\rho, \widetilde{z})\right| \\ &\leqslant \left|\left((\widetilde{B} - B)e, \widetilde{z}\right)\right| + \left|(\rho, \widetilde{z} - \widetilde{z}_h)\right| + \sum_{k=1}^{n} \left|\widetilde{z}_{h,k}\right| \cdot \left|\rho_k\right|. \end{aligned} \tag{1.19}$$

Remark

$$\begin{aligned} \|\widetilde{B} - B\| &\leq \left\| \int_0^1 \left\{A'(u_h) - A'(u_h + se)\right\} ds \right\| \\ &\leqslant \frac{1}{2} \cdot L' \cdot \|e\|, \end{aligned}$$

where

$$L' = \sup_{(u_h, s, e)} \left|A''\right|$$

is an upper bound on the second derivatives of the components of A. Then, equation

(1.19) becomes

$$|J(e)| \leqslant \frac{1}{2} \cdot L' \cdot \|e\|^2 \cdot \|\widetilde{z}\| + \|\rho\| \cdot \|\widetilde{z} - \widetilde{z}_h\| + \sum_{k=1}^{n} \left|\widetilde{z}_{h,k}\right| \cdot \left|\rho_k\right|. \tag{1.20}$$

Since the first two items are very small as $h \to 0$, the last one dominates the right side of (1.20). So, (1.20) is a kind of a posteriori analysis. This shows that the adjoint method can estimate the error in a functional of the approximation.

In conclusion, we see that a posteriori error estimation requires solving an adjoint problem that depends on the continuous differential operator. This slightly more complex formulation has significant advantages. First of all, it allows an estimation of the error without knowledge of the exact solution. Only the approximate solution is used in the estimation of the error and in this sense, a posteriori error estimation is constructive. Secondly, a posteriori error estimation allows one to estimate the error with respect to a functional of the solution rather than with respect to some gross norm of the solution. The resulting error estimates therefore allow optimal mesh adaptation with respect to precise engineering objectives.

1.2 Conservation Laws

In this thesis, we study error estimation of numerical solutions of conservation laws. So, we need to describe conservation laws. We provide a few examples and some basic theorems concerning conservation laws.

1.2.1 Two examples of conservation laws

Example #1 Consider gas in a tube. Let $\rho(x,t)$ and $v(x,t)$ be the density and velocity of the gas at point (x,t). So, at time t, the mass in $[x_1, x_2]$ is $\int_{x_1}^{x_2} \rho(x,t)dx$ and the mass flux for each point (x,t) is $\rho(x,t)v(x,t)$. Suppose the tube is sealed, then the mass in the tube cannot be lost, i.e. the mass is conserved and we get the *integral form* of the conservation laws

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho(x,t)dx = \rho(x_1,t)v(x_1,t) - \rho(x_2,t)v(x_2,t). \tag{1.21}$$

Now, if the time $t \in [t_1, t_2]$, then, after integrating both sides of (1.21), we get

$$\begin{aligned} \int_{x_1}^{x_2} \rho(x,t_2) - \int_{x_1}^{x_2} \rho(x,t_1)dx \\ = \int_{t_1}^{t_2} \rho(x_1,t)v(x_2,t)dt - \int_{t_1}^{t_2} \rho(x_2,t)v(x_2,t)dt. \end{aligned} \tag{1.22}$$

If $\rho(x,t)$ and $v(x,t)$ are differentiable, using

$$\begin{aligned} \rho(x,t_2) - \rho(x,t_1) &= \int_{t_1}^{t_2} \frac{\partial}{\partial t}\rho(x,t)\,dt \\ \rho(x_2,t)v(x_2,t) - \rho(x_1,t)v(x_1,t) &= \int_{x_1}^{x_2} \frac{\partial}{\partial x}\rho(x,t)v(x,t)\,dx \end{aligned}$$

in (1.22), we get

$$\int_{t_1}^{t_2}\int_{x_1}^{x_2} \left\{\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x}\big(\rho(x,t)v(x,t)\big)\right\} dxdt = 0.$$

Since this equation must hold for any domain $\Omega := \{(x,t) : x_1 \leqslant x \leqslant x_2, t_1 \leqslant t \leqslant t_2\}$, we can get the *differential form* of the conservation laws

$$\rho_t + (\rho v)_x = 0. \tag{1.23}$$

Example #2 Consider traffic flow on a highway. Let $\rho(x,t)$ and $v(x,t)$ be the density and velocity of cars. Suppose the number of cars doesn't change, then the density and velocity must have a relationship

$$\rho_t + (\rho v)_x = 0. \tag{1.24}$$

Suppose v is a given function of ρ such that we can obtain a scalar conservation for ρ alone. Physically this means that: on a highway, we would like to drive as fast as we can, but in heavy traffic we have to slow down, with velocity decreasing as density increases. For simplicity, we assume that there is a linear relationship between ρ and v

$$v(\rho) = v_{\max}(1 - \rho/\rho_{\max}).$$

Here, the velocity v is a function of ρ, not a constant, so we rewrite (1.24) as

$$\rho_t + f(\rho)_x = 0 \tag{1.25}$$

where

$$f(\rho) = \rho v_{\max}(1 - \rho/\rho_{\max}).$$

1.2.2 Two forms of conservation laws

From both examples above, we know that there are two forms of conservation laws: integral and differential forms. Following the presentation of Randall J.LeVeque (Randall, 1992), we suppose $u = u(x,t)$ is the solution of the Cauchy problem:

$$\begin{aligned} u_t + f(u)_x &= 0 \qquad (1.26) \\ u(x,0) &= u_0(x). \end{aligned}$$

If u is an intensive quantity, then $U(t) = \int_{x_1}^{x_2} u(x,t)\,dx$ is the total quantity of the extensive variable within the interval $[x_1, x_2]$ at time t. Since the extensive quantity is conserved, we obtain the *integral form* of the conservation laws:

$$\frac{d}{dt}\big(U(t)\big) = \frac{d}{dt}\int_{x_1}^{x_2} u(x,t)\,dx = f\big(u(x_2,t)\big) - f\big(u(x_1,t)\big).$$

As in Example#1 above, we can get the differential form from the integral form of the conservation laws:

$$u_t + f(u)_x = 0. \qquad (1.27)$$

The integral form can be used even for discontinuous solutions while the differential form is only used for smooth solutions. But, the integral form is more difficult to handle than the differential form. Generally, we can work with the differential form with "jump conditions" which come from the integral form.

1.2.3 Smooth solutions

In the thesis, we consider the conservation law with initial value, i.e. Cauchy problem:

$$u_t + f(u)_x = 0, \qquad (1.28)$$
$$u(\cdot, 0) = u_0(\cdot).$$

First, we consider properties of smooth solutions to conservation law.

Case #1 Suppose the flux $f(u)$ in (1.26) can be written as $f(u) = au$, where a is a constant. We can rewrite (1.26) as

$$\begin{aligned} u_t + au_x &= 0 \\ u(x,0) &= u_0(x), \end{aligned}$$

here, the characteristics satisfy the equations

$$\begin{aligned} x'(t) &= a \\ x(0) &= x_0. \end{aligned}$$

So, we find that along these characteristics $x(t)$,

$$\begin{aligned} \frac{d}{dt}u\big(x(t),t\big) &= \frac{\partial}{\partial t}u\big(x(t),t\big) + \frac{\partial}{\partial x}\Big(u\big(x(t),t\big)\Big)x'(t) \\ &= u_t + au_x \\ &= 0. \end{aligned}$$

Therefore, the value of u is constant along their characteristics and the general solution $u(x,t) = u_0(x - at)$.

Case #2 Suppose the flux $f(u)$ in (1.26) is a nonlinear function, like the Example #2, we can rewrite (1.26) as

$$u_t + f'(u) \cdot u_x = 0$$
$$u(x,0) = u_0(x).$$

Here, the characteristics satisfy the equations

$$x'(t) = f'(u)$$
$$x(0) = x_0.$$

So, we find

$$\begin{aligned}
\frac{d}{dt}u(x(t),t) &= \frac{\partial}{\partial t}u(x(t),t) + \frac{\partial}{\partial x}u(x(t),t)x'(t) \\
&= u_t + f'(u) \cdot u_x \\
&= 0,
\end{aligned}$$

Again, the value of u is constant along their characteristics. But since the value of $f'(u)$ is not constant, so these characteristics may cross each other when time increases, i.e. the solution u may become discontinuous even though the initial value u_0 is continuous. When it happens, the solution is not unique anymore and we need to give an additional condition to find the correct solution. For this reason, we need to introduce solutions with discontinuities.

1.2.4 Solutions with discontinuities

In order to define the solution of conservation laws beyond the time when discontinuities first appear, we present some definitions and theorems which are described in the book by Bressan (Bressan, 2001).

Definition 1.1 *We say that f is Lipschitz continuous if $|f(x) - f(y)| \leqslant L|x-y|$ for some constant L (Lipschitz constant) and all x, y in the domain of f.*

Definition 1.2 *The total variation of u on the interval $[a, b]$ is*

$$TV\{u\} \doteq \sup_{x_0 < x_1 < \cdots < x_N} \left\{ \sum_{i=1}^{N} |u(x_i) - u(x_{i-1})| \right\}, \tag{1.29}$$

where the supremum is taken over all subdivision $a = x_0 < x_1 < \cdots < x_N = b$. If the $TV\{u\}$ is bounded, then we say that u has bounded variation which is written as $u \in BV$ $([a, b])$.

Definition 1.3 *Let Ω be the domain of a function $u(x)$, then*

$$\text{if } \int_\Omega |u(x)|dx < \infty, \qquad \text{then } u \in L^1(\Omega)$$
$$\text{if } \max |u| < \infty, \qquad \text{then } u \in L^\infty.$$

In practice, we say that $\frac{\partial u}{\partial x} \in L^\infty$ if there exists $M (\in \mathbb{R})$ such that

$$\left| \frac{u(x+h) - u(x)}{h} \right| \leqslant M \qquad \text{for all } x \text{ and } h.$$

If $u \in L^1$ *and* $\frac{\partial u}{\partial x} \in L^\infty$, *then we say that* $u \in W^{1,\infty}$.

Definition 1.4 *A measurable function* $u = u(x,t)$ *from an open set* $\Omega \subseteq \mathbb{R} \times \mathbb{R}$ *is a distributional solution to the equation of conservation law*

$$u_t + f(u)_x = 0, \tag{1.30}$$

if, for every $W^{1,\infty}$ *function* ϕ*:* $\Omega \mapsto \mathbb{R}$ *with compact support, one has*

$$\int\int_\Omega \{u\phi_t + f(u)\phi_x\}\, dxdt = 0. \tag{1.31}$$

Definition 1.5 *Given an initial condition*

$$u(x,0) = u_0(x), \tag{1.32}$$

then the function u *is a weak solution of the problem* (1.30), (1.32) *if* u *is continuous as a function from* $[0,T]$ *into* L^1_{loc}, *the initial condition* (1.32) *holds and the restriction of* u *to the open strip* $]0,T[\times\mathbb{R}$ *is a distributional solution of* (1.30).

We see that there is no continuity assumption for u here. We only need that u and $f(u)$ should be locally integrable in the domain Ω. Now, we can consider possibly discontinuous solutions u and we now attempt to define a unique weak solution according to this definition.

As shown in Figure 1.1, we suppose that u has a discontinuity along the curve $x(t)$. Let the left side of the discontinuity be u_l, $f(u_l)$, and the right side be u_r, $f(u_r)$. For shock

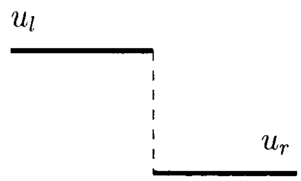

Figure 1.1 Discontinuity in u.

wave, the speed of discontinuity S should satisfy **Rankine-Hugoniot jump condition**

$$S = \frac{f(u_l) - f(u_r)}{u_l - u_r} = \frac{[f]}{[u]}. \tag{1.33}$$

Then we say that $u(x,t)$ is the *entropy solution* if all discontinuities satisfy the *entropy condition*:

$$\frac{f(u) - f(u_l)}{u - u_l} \geqslant S \geqslant \frac{f(u) - f(u_r)}{u - u_r} \qquad \text{for all } u \in (u_l, u_r). \tag{1.34}$$

This condition guarantees that characteristics move into the shock wave. This is equivalent, in many cases, to the second law of thermodynamics (Dafermos, 2000). This is also equivalent, for strictly convex fluxes of the type we will consider, to the following definition.

Definition 1.6 *A measurable function* $u = u(x,t)$ *from an open set* $\Omega \subseteq \mathbb{R} \times \mathbb{R}$ *is said to satisfy the weak form of the entropy condition to the conservation law* (1.30) *if, for every pair* (U, V) *satisfying* U *strictly convex and* $V' = f'U'$*, the inequality*

$$\int\int_\Omega \{U(u)\phi_t + V(u)\phi_x\}\, dxdt \geq 0. \tag{1.35}$$

holds for every $W^{1,\infty}$ *function* ϕ*:* $\Omega \mapsto \mathbb{R}$ *with compact support.*

For the Example #2, suppose

$$\rho(x,0) = \begin{cases} \rho_l & x < 0, \\ \rho_r & x > 0 \end{cases},$$

where $0 < \rho_l < \rho_r < \rho_{max}$. Then, from (1.33), we know that the shock speed S should be

$$\begin{aligned} S &= \frac{f(\rho_l) - f(\rho_r)}{\rho_l - \rho_r} \\ &= u_{max}\big(1 - (\rho_l + \rho_r)/\rho_{max}\big). \end{aligned}$$

Assume $\rho_r = \rho_{max}$, then $S < 0$ and the shock will propagate to the left. It makes sense: a man drives a car at speed u_l before he encounters a traffic jam, he reduces his speed to 0 while the density jumps from ρ_l to ρ_{max}. This discontinuity occurs at the shock, and clearly the shock location moves behind the car as it joins the traffic jam.

We now mention the fundamental existence, uniqueness and stability of BV solutions for scalar conservation laws (Dafermos, 2000; Bressan, 2001; LeFloch, 2002).

Theorem 1.7 *For the Cauchy problem, let f be locally Lipschitz continuous and $u_0 \in L^1$ have bounded variation. Then there is an entropy weak solution $u = u(x,t)$, with*

$$TV\{u(\cdot,t)\} \leqslant TV\{u_0\}, \ \ \|u(\cdot,t)\|_{L^\infty} \leqslant \|u_0\|_{L^\infty} \qquad \text{for all } t \geqslant 0. \tag{1.36}$$

Suppose v is also an entropy weak solution, let M, L be constants such that

$$|u(x,t)| \leqslant M,\ |v(x,t)| \leqslant M \qquad \textit{for all } t, x,$$

$$|f(w) - f(w')| \leqslant L|w - w'| \qquad \textit{for all } w, w' \in [-M, M].$$

Then, for every $R > 0$ and $\tau \geqslant \tau_0 \geqslant 0$, we have

$$\int_{|x| \leqslant R} |u(x,\tau) - v(x,\tau)|dx \leqslant \int_{|x| \leqslant R + L(\tau - \tau_0)} |u(x,\tau_0) - v(x,\tau_0)|dx. \tag{1.37}$$

1.3 Previous Results

In this section, we recollect briefly the development of the error estimation that we will describe below. We also present some references to the extensive literature on error estimations that are of particular relevance for our work.

In the early last century, several people produced error estimations. For example, L.F. Richardson developped first the error estimation for ordinary differential equations: Richardson extrapolation is used to improve the rate of convergence of a sequence (Richardson, 1910; Richardson, 1927). Later in the 60's, Erwin Fehlberg developped another error estimation for ordinary differential equations: Runge-Kutta-Fehlberg method which can get an error estimator by taking the difference of a high and a low order Runge-Kutta method (Fehlberg, 1969). In the 70's, Ivo Babuška and W. C. Rheinholt introduced the theory of a posteriori error estimation for linear elliptic problems (Babuška and Rheinboldt, 1978) and (Babuška and Rheinboldt, 1979). This is now widely used in engineering computations. In the 80's, Babuška and his colleagues extended their work on a posteriori error estimates, and most of his estimates were obtained with the help of the adjoint method. During that time, Babuška also studied some other methods, such as superconvergent patch recovery technique proposed by Zienkiewicz and Zhu (Babuška et al., 1997). In 2000, J. Tinsley Oden and M. Ainsworth published a book which summarized many methods for a posteriori estimator for finite element approximations of partial differential equations especially for scalar elliptic problems in two-dimensional domains, such as the least squares error estimator, subdomain residual method and element residual method, Zienkiewicz-Zhu patch recovery estimator and equilibrated residual method (Ainsworth and Oden, 2000). Many of these methods for elliptic equations, are now used in production and commercially distributed codes.

Error estimations by the adjoint method is not as well developped for hyperbolic problems. Eitan Tadmor (Tadmor, 1991) used the adjoint method to obtain a stability estimate for approximations to the entropy solution of a convex scalar conservation law. For linear hyperbolic problems, the adjoint method of error estimation for finite element methods has been developped in large part by Johnson, Rannacher and Süli. Johnson's initial work focused on SDFEM (Johnson, 1993). He also worked on linear & nonlinear parabolic problems with space-time Galerkin methods (Eriksson and Johnson, 1995b; Eriksson and Johnson, 1995a). He worked on a posteriori error estimation for Euler's equations (Johnson and Szepessy, 1995). In this important paper, he studied a posteriori error bound for approximations of one-dimensional systems of nonlinear conservation laws with shocks, but he supposed all these shocks are weak and don't interact with each other. Rannacher and Becker developped a general formulation for weighted a posteriori error estimators based on adjoint problems (Becker and Rannacher, 2001). He also studied nonlinear functionals but nonlinear PDEs were treated by linearization. Endre Süli studied a lot about a posteriori error analysis in Galerkin finite element approximations for hyperbolic systems (Süli and Houston, 1996) and (Süli, 1999). However, his study was applied only to linear systems.

For hyperbolic problems, other people have tried methods not based on the adjoint problem. In particular, for finite difference approximations, we mention the following. Gosse and Makridakis proposed two a posteriori error estimates for one-dimensional scalar conservation laws. Unfortunately, their estimates were not optimal (Gosse and Makridakis, 2000). Mario Ohlberger developped several a posteriori error estimates for approximate solutions to time-dependent nonlinear scalar conservation laws in n-D built with finite volume approximations (Kröner and Ohlberger, 2000; Ohlberger, 2001; Kröner et al., 2003). Those results were based on Kruskov's stability theory. Starting in the late 80's, Joseph E. Flaherty and his colleagues worked on a posteriori error estimation in a finite element method for one- and two- dimensional parabolic systems (Adjerid and Flaherty, 1988) and for one- and two-dimensional hyperbolic systems. Actually, they apply their results to nonlinear conservation laws but mathematically they measured only to the spatial error and they assume the results are smooth.

As far as rigorous studies of the adjoint problems are concerned, we mention the first work of Tadmor (Tadmor, 1991). He showed the existence of a solution to the adjoint problems. Bouchut and James showed that, for scalar conservation laws, there exists a unique solution to the adjoint by picking the one with the smallest BV among all possible solutions (Bouchut and James, 1998). This BV condition will be used in this thesis. Oleinik demonstrated uniqueness to solutions of 2×2 systems of conservation laws by constructing a solution to the adjoint problem. In the L^2 framework, LeFloch and Xin (LeFloch and Xin, 1993) have demonstrated existence of the adjoint but this cannot be used in the standard L^1, BV framework. Besides these results, there are very few results for systems of conservation laws.

The focus of this thesis is to provide a rigorous application of the adjoint method. We study the front-tracking method since it is the only scheme, besides Glimm's scheme, for which we possess a stability theory for systems of nonlinear conservation laws. It is therefore hoped that this approach can be extended to systems of conservation laws. The front-tracking method has been used to study the existence of solutions to conservation laws with discontinuous fluxes (Gimse and Risebro, 1992; Klausen and Risebro, 1999; Klingenberg and Risebro, 1995) but error estimation with the adjoint method has never been addressed using this technique.

CHAPTER 2

FRONT-TRACKING METHOD

Consider the Cauchy problem

$$u_t + f(u)_x = 0, \tag{2.1}$$
$$u(\cdot, 0) = u_0(\cdot),$$

where $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is smooth with n real and distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ for $Df(u)$ and all $u \in \Omega$. A fundamental result of Glimm (Glimm, 1965) states that if $u_0 \in L^1(\mathbb{R})$ and $TV(u_0)$ is sufficiently small then there exists an entropy solution u, in the sense of Lax, belonging to

$$W^{1,\infty}_{\text{loc}}\big([0,T], L^1_{\text{loc}}(\mathbb{R}^n)\big) \cap L^{\infty}_{\text{loc}}\big([0,T], \text{BV}_{\text{loc}}(\mathbb{R}^n)\big). \tag{2.2}$$

Recent results of Bressan, Crasta and Piccoli (Crasta et al., 2000), LeFloch and Hu (Hu and LeFloch, 2001) and of Liu and Yang (Liu and Yang, 1999) show that the solution generated by Glimm's (Glimm, 1965) scheme or the front-tracking method is unique within the class of entropy solutions. We will use the front tracking method to generate a sequence of approximate solutions u^ϵ that converges to u in this space as $\epsilon \longrightarrow 0$.

2.1 Front-tracking approximations for scalar convex conservation law

The thesis is concerned with the Cauchy problem for a scalar convex conservation law, i.e. $n = 1$ in (2.1). All solutions are obtained by using the front-tracking method. So, we need to explain this method briefly.

The front-tracking method is close to Glimm's scheme (Glimm, 1965) and to Liu's wave-tracing extension (Liu, 1977). For scalar conservation laws, it was first proposed by Dafermos (Dafermos, 1972) and later extended to systems of nonlinear conservation laws by DiPerna, Risebro and Bressan. This method produces a simple piecewise constant approximation formed entirely of travelling discontinuities. The goal of this section is to prove the following Theorem.

Theorem 2.1 (Dafermos) *If $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $f \in C^2(\mathbb{R})$ and is strictly convex, then the front-tracking approximation converges as $\epsilon \to 0$ to the entropy solution of the Cauchy problem.*

For each positive $\epsilon > 0$, choose a piecewise constant function u_0^ϵ with a finite number of discontinuities that satisfies

$$||u_0 - u_0^\epsilon||_{L^1} < \epsilon,$$

$$TV(u_0^\epsilon) \leqslant TV(u_0).$$

To define $u^\epsilon(\cdot, t)$ for all $t \geqslant 0$ that is piecewise constant in space, it suffices to define $u^\epsilon(\cdot, 0) = u_0^\epsilon$ and define the propagation of each discontinuity.

Suppose there is a discontinuity α located at $x_\alpha(t)$. If the left and right hand states

$$u_\alpha^-(t) = \lim_{x \to x_\alpha^-(t)} u^\epsilon(x,t),$$

$$u_\alpha^+(t) = \lim_{x \to x_\alpha^+(t)} u^\epsilon(x,t),$$

are such that $u_\alpha^-(t) > u_\alpha^+(t)$, then we call α a *shock wave.* We suppose that the speed of propagation satisfies

$$\left|\dot{x}_\alpha(t) - S(u_\alpha^-(t), u_\alpha^+(t))\right| < \epsilon,$$

where S is defined in the formula (1.33). If $0 < u_\alpha^+(t) - u_\alpha^-(t) \leqslant \epsilon$, then we call α a *rarefaction wave.* If $0 < \epsilon < u_\alpha^+(t) - u_\alpha^-(t)$, then we split the discontinuity into k discontinuities separated by

$$u_\alpha^-(t) = u^{(0)} < u^{(1)} < u^{(2)} < \cdots < u^{(k)} = u_\alpha^+(t)$$

so that $u^{(i)} - u^{(i-1)} < \epsilon$ and that each discontinuity at $y^{(i)}(t)$ with left and right hand states $u^{(i-1)}$ and $u^{(i)}$ satisfies

$$\left|\dot{y}_\alpha^{(i)}(t) - S(u^{(i-1)}, u^{(i)})\right| < \epsilon.$$

We also suppose that no two neighbouring rarefaction waves cross paths. This second hypothesis is very easy to satisfy since the flux is convex and therefore the rarefaction shock in a pair of neighboring rarefaction shocks travels at a speed less than or equal to the speed of it's neighbor (we are trying to avoid the case where these speeds are equal and thereby generate an infinite number of interactions).

The resulting algorithm has been used in some numerical applications and been found to be relatively straightforward to implement (Risebro and Tveito, 1992). The algorithm, apart from the Riemann solver, appears to be essentially of the same cost as a typical first order finite difference scheme. Adaptive versions of this scheme also exist, based on heuristic error estimators (Lie et al., 1998).

To prove that the approximations u^ϵ converge, we will use Helly's Theorem.

Theorem 2.2 (Helly) *Consider a sequence of functions* u^ϵ*:* $[0,\infty)\times\mathbb{R}\mapsto\mathbb{R}^n$ *with the following properties:*

$$TV\big(u^\epsilon(\cdot,t)\big)\leqslant C,\qquad |u^\epsilon(x,t)|\leqslant M\qquad \textit{for all } t,x,$$
$$\int_{-\infty}^{\infty}|u^\epsilon(x,t)-u^\epsilon(x,s)|\,dx\leqslant L|t-s|\qquad \textit{for all } t,s\geqslant 0,$$

for some constants C*,* M*,* L*. Then there exists a subsequence* $\{u^\epsilon\}_\epsilon$ *which converges to some function* u *in* $L^1_{loc}([0,\infty)\times\mathbb{R};\mathbb{R}^n)$*. This limit function satisfies*

$$\int_{-\infty}^{\infty}|u(x,t)-u(x,s)|\,dx\leqslant L|t-s|\qquad \textit{for all } t,s\geqslant 0.$$

The point values of the limit function u *can be uniquely determined by requiring that*

$$u(x,t)=u(x+,t)\doteq\lim_{y\to x+}u(y,t)\qquad \textit{for all } t,x.$$

In this case, one has

$$TV\big(u(\cdot,t)\big)\leqslant C,\qquad |u(x,t)|\leqslant M\qquad \textit{for all } t,x.$$

Proof We will demonstrate Theorem 2.1 by roughly following the ideas of Bressan (Bressan, 2001).

Proof of Theorem 2.1 The first step is to show the three bounds in Helly's Theorem. At time t, let $\mathcal{W}(u^\epsilon(t))$ be the set of all discontinuities in $u^\epsilon(\cdot, t)$ and define

$$TV\big(u^\epsilon(t)\big) = \sum_{\alpha \in \mathcal{W}(u^\epsilon(t))} |\sigma_\alpha|.$$

where $\sigma_\alpha = u_\alpha^+(t) - u_\alpha^-(t)$ is the strength of wave α and is constant between interactions in u^ϵ. Suppose a discontinuity α crosses a discontinuity β, then

Case 1: if α and β are shocks, we suppose the outgoing discontinuity is a shock wave of strength $\sigma_\alpha + \sigma_\beta$,

Case 2: if α is a shock and β is a rarefaction (or vice versa) then either

$a)$ the result is a shock when $\sigma_\alpha + \sigma_\beta < 0$,

$b)$ the result is a rarefaction when $\sigma_\alpha + \sigma_\beta \geqslant 0$.

Nevertheless, if it is a rarefaction $\sigma_\alpha + \sigma_\beta < \sigma_\beta < \epsilon$.

If an interaction occurs at time $t^{(i)}$, then in the Case 1

$$TV\big(u^\epsilon(t^{(i)}-)\big) = TV\big(u^\epsilon(t^{(i)}+)\big),$$

but in the Case 2

$$TV\big(u^\epsilon(t^{(i)}-)\big) = TV\big(u^\epsilon(t^{(i)}+)\big) + C_{\alpha\beta}$$

where the cancellation is

$$C_{\alpha\beta} = \Big| |\sigma_\alpha| + |\sigma_\beta| \Big| - \Big| |\sigma_\alpha| - |\sigma_\beta| \Big| > 0.$$

The cancellation is non-zero specifically when a shock meets a rarefaction. In general, the shock will be strong and the rarefaction weak so that after interaction, the strength of the outgoing shock σ_α will have been decreased by $-2\sigma_\beta$, β being the rarefaction, by the interaction. In conclusion, it is obvious that

$$TV\big(u^\epsilon(t)\big) \leqslant TV\big(u_0^\epsilon\big) \leqslant TV\big(u_0\big). \tag{2.3}$$

A similar analysis also shows that

$$\big\|u^\epsilon(\cdot,t)\big\|_{L^\infty(\mathbb{R})} \leqslant \big\|u_0^\epsilon\big\|_{L^\infty(\mathbb{R})} \leqslant \big\|u_0\big\|_{L^\infty(\mathbb{R})}, \tag{2.4}$$

because the maximum value of the discontinuity will be reduced during each interaction while the minimum value can only increase.

Suppose δ is small enough that no discontinuities cross during the time interval $(t, t+\delta)$. This is possible because at any time t, the number of discontinuities in u^ϵ is finite (in fact, it decreases in time). Since the value of u^ϵ is constant between any two neighbouring discontinuities (travelling at constant speeds), we get

$$
\begin{aligned}
\|u^\epsilon(x,t+\delta) - u^\epsilon(x,t)\|_{L^1(\mathbb{R})} &= \int_{-\infty}^{+\infty} |u^\epsilon(x,t+\delta) - u^\epsilon(x,t)| dx \\
&= \sum_{\alpha \in \mathcal{W}(u^\epsilon(t))} |\sigma_\alpha| \cdot |x_\alpha(t+\delta) - x_\alpha(t)| \\
&= \sum_{\alpha \in \mathcal{W}(u^\epsilon(t))} |\sigma_\alpha| \cdot |\dot{x}_\alpha(\xi)| \cdot |\delta| \\
&\leqslant C \cdot |\delta| \cdot \sum_{\alpha \in \mathcal{W}(u^\epsilon(t))} |\sigma_\alpha| \\
&= C \cdot TV(u_0) \cdot |\delta|, \qquad (2.5)
\end{aligned}
$$

where $C = \max |\dot{x}_\alpha|$. Obviously, this continues to be true for larger δ.

Helly's Theorem states that because (2.3), (2.4) and (2.5) hold, then there exists a subsequence of $\{u^\epsilon\}_\epsilon$ that converges to a function $u : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ in $L^1_{\text{loc}}([0,\infty) \times \mathbb{R}, \mathbb{R})$ that satisfies

$$
\begin{aligned}
\|u(\cdot,t)\|_{L^\infty(\mathbb{R})} &\leqslant \|u_0\|_{L^\infty(\mathbb{R})} \\
TV(u(\cdot,t)) &\leqslant TV(u_0) \\
\int_{\mathbb{R}} |u^\epsilon(x,t) - u^\epsilon(x,s)| dx &\leqslant L \cdot |t-s|.
\end{aligned}
$$

Now, we need to show that u^ϵ is an entropy solution of the conservation law (2.1), that is to say that it satisfies

$$
I = \int \int \{|u^\epsilon - k|\phi_t + (f(u^\epsilon) - f(k))\text{sgn}(u^\epsilon - k)\phi_x\} dxdt \geqslant 0, \qquad (2.6)
$$

for every $k \in \mathbb{R}$ and every positive ϕ with compact support. As shown in (Bressan, 2001), this is equivalent to satisfying all of the Kruškov entropies which then implies the general inequality (1.35) for all convex entropies.

Let K = support $\{\phi\}$, then by Egoroff's theorem (Folland, 1984), there exists a subset $K_\epsilon \subset K$ such that

$$i) \quad \text{area}(K \setminus K_\epsilon) < C \cdot \epsilon,$$

$$ii) \quad \text{sgn}\big(u^\epsilon(x,t) - k\big) = \text{sgn}\big(u(x,t) - k\big) \qquad \forall x, t \in K_\epsilon.$$

This emphasizes that the domain where $\text{sgn}\big(u^\epsilon(x,t)-k\big) \neq \text{sgn}\big(u(x,t)-k\big)$ is too small to consider as $\epsilon \longrightarrow 0$. Now, let us consider ϕ with compact support into small ball B. It suffices to show (2.6) when the support of ϕ is limited to a small ball B. If the B is very small, since u^ϵ has a finite number of discontinuities, then we can assume that there is only one discontinuity, so the ball B is separated into two half balls B_i, where $i = \{1, 2\}$ and the common boundary is Q. If the discontinuity α has left and right states as $u_l^\epsilon, u_r^\epsilon$ with norm $n_1 = (1, -\dot{x}_\alpha(t)) \cdot \lambda$ and $n_2 = (-1, \dot{x}_\alpha(t)) \cdot \lambda$, where $\lambda = \frac{1}{\sqrt{1+(\dot{x}_\alpha(t))^2}}$, then we have

$$\begin{aligned}
&\int\int_B \{|u^\epsilon - k|\phi_t + (f(u^\epsilon) - f(k))\text{sgn}(u^\epsilon - k)\phi_x\}dxdt \\
&\quad = \sum_{i=1}^{2} \int\int_{B_i} \big(|u^\epsilon - k|, (f(u^\epsilon) - f(k))\text{sgn}(u^\epsilon - k)\big) \cdot \nabla\phi dxdt \\
&\quad = \sum_{i=1}^{2} \int\int_{B_i} \Big\{\frac{\partial}{\partial t}\big(|u^\epsilon - k| \cdot \phi\big) + \frac{\partial}{\partial x}\big((f(u^\epsilon) - f(k))\text{sgn}(u^\epsilon - k) \cdot \phi\big)\Big\}dxdt \\
&\quad = \sum_{i=1}^{2} \int_{\partial B_i} \Big[|u^\epsilon - k| \cdot \phi \cdot n_1 + (f(u^\epsilon) - f(k))\text{sgn}(u^\epsilon - k) \cdot \phi \cdot n_2\Big] dQ
\end{aligned}$$

$$
\begin{aligned}
&= \int_Q \Big\{ |u^\epsilon - k| \cdot \phi \cdot \big(- \dot{x}_\alpha(t) \cdot \lambda \big) \\
&\qquad\qquad + (f(u^\epsilon) - f(k)) \mathrm{sgn}(u^\epsilon - k) \cdot \phi \cdot 1 \cdot \lambda \Big\} dQ \\
&\quad + \int_Q \Big\{ |u^\epsilon - k| \cdot \phi \cdot \big(\dot{x}_\alpha(t) \cdot \lambda \big) \\
&\qquad\qquad + (f(u^\epsilon) - f(k)) \mathrm{sgn}(u^\epsilon - k) \cdot \phi \cdot (-1) \cdot \lambda \Big\} dQ \\
&= \int_Q \phi \cdot \lambda \cdot \Big\{ \Big[|u_l^\epsilon - k| - |u_r^\epsilon - k| \Big] \cdot \big(- \dot{x}_\alpha(t) \big) \\
&\quad + \Big[(f(u_l^\epsilon) - f(k)) \mathrm{sgn}(u_l^\epsilon - k) - (f(u_r^\epsilon) - f(k)) \mathrm{sgn}(u_r^\epsilon - k) \Big] \Big\} dQ
\end{aligned}
\tag{2.7}
$$

If $u_l^\epsilon, u_r^\epsilon \geqslant k$ or $u_l^\epsilon, u_r^\epsilon \leqslant k$, then the (2.7) becomes

$$
\begin{aligned}
I &= \int_Q \phi \cdot \lambda \cdot \mathrm{sgn}(u^\epsilon - k) \cdot \Big\{ - \Big[(u_l^\epsilon - k) - (u_r^\epsilon - k) \Big] \cdot \dot{x}_\alpha(t) \\
&\qquad + \Big[\big(f(u_l^\epsilon) - f(k) \big) - \big(f(u_r^\epsilon) - f(k) \big) \Big] \Big\} dQ \\
&= \int_Q \phi \cdot \lambda \cdot \mathrm{sgn}(u^\epsilon - k) \cdot \Big\{ - \Big[(u_l^\epsilon - u_r^\epsilon) \cdot s(u_l^\epsilon, u_r^\epsilon) + \big(f(u_l^\epsilon) - f(u_r^\epsilon) \big) \Big] \\
&\qquad - (u_l^\epsilon - u_r^\epsilon) \cdot \big(\dot{x}_\alpha - s(u_l^\epsilon, u_r^\epsilon) \big) \Big\} dQ.
\end{aligned}
$$

Remark we regard u^ϵ as a convex entropy with entropy flux $f(u^\epsilon)$, so we have

$$
\begin{aligned}
I &= \int_Q \phi \cdot \lambda \cdot \mathrm{sgn}(u^\epsilon - k) \cdot \big(- (u_l^\epsilon - u_r^\epsilon) \big) \cdot \big(\dot{x}_\alpha - s(u_l^\epsilon, u_r^\epsilon) \big) \, dQ \\
&= \int_Q \phi \cdot C \cdot \epsilon \; dQ,
\end{aligned}
$$

where $C = \max\{\mathrm{sgn}(u^\epsilon - k) \cdot (u_r^\epsilon - u_l^\epsilon) \cdot \lambda\}$. It is clear that $I \to 0$ as $\epsilon \to 0$.

If the discontinuity α is with k between a shock interval $[u_l^\epsilon, u_r^\epsilon]$, then the (2.7) becomes

$$\begin{aligned}
I &= \int_Q \phi \cdot \lambda \cdot \Big\{ \big[(u_l^\epsilon - k) + (u_r^\epsilon - k) \big] \cdot \big(- \dot{x}_\alpha(t) \big) \\
&\quad + \big[\big(f(u_l^\epsilon) - f(k) \big) + \big(f(u_r^\epsilon) - f(k) \big) \big] \Big\} dQ \\
&= \int_Q \phi \cdot \lambda \cdot \Big\{ \big[u_l^\epsilon + u_r^\epsilon - 2k \big] \cdot \big(- \dot{x}_\alpha(t) \big) \\
&\qquad + \big[f(u_l^\epsilon) + f(u_r^\epsilon) - 2f(k) \big] \Big\} dQ \\
&= \int_Q \phi \cdot \lambda \cdot \Big\{ \big[\big(u_l^\epsilon + u_r^\epsilon - 2k \big) \cdot \big(- s(u_l^\epsilon, u_r^\epsilon) \big) + \big(f(u_l^\epsilon) + f(u_r^\epsilon) - 2f(k) \big) \big] \\
&\qquad - \big[u_l^\epsilon + u_r^\epsilon - 2k \big] \cdot \big(\dot{x}_\alpha - s(u_l^\epsilon, u_r^\epsilon) \big) \Big\} dQ. \qquad (2.8)
\end{aligned}$$

Now, let

$$g(k) = -[u_l^\epsilon + u_r^\epsilon - 2k] \cdot s(u_l^\epsilon, u_r^\epsilon) + [f(u_l^\epsilon) + f(u_r^\epsilon) - 2f(k)].$$

Clearly we have

$$\begin{aligned}
g(u^{\epsilon_l}) &= -[-u_l^\epsilon + u_r^\epsilon] \cdot s(u_l^\epsilon, u_r^\epsilon) + [-f(u_l^\epsilon) + f(u_r^\epsilon)] = 0 \\
g(u^{\epsilon_r}) &= -[u_l^\epsilon - u_r^\epsilon] \cdot s(u_l^\epsilon, u_r^\epsilon) + [f(u_l^\epsilon) - f(u_r^\epsilon)] = 0 \\
g''(k) &= -2 \cdot f''(k) < 0,
\end{aligned}$$

so $g(k)$ must be positive for all $k \in [u_l^\epsilon, u_r^\epsilon]$, therefore, the I in (2.8) becomes

$$\begin{aligned}
I &= \int_Q \phi \cdot \lambda \cdot \big[g(k) - (u_l^\epsilon + u_r^\epsilon - 2k) \cdot \big(\dot{x}_\alpha - s(u_l^\epsilon, u_r^\epsilon) \big) \big] dQ \\
&= \int_Q \phi \cdot \lambda \cdot g(k) + \phi \cdot C \cdot \epsilon \, dQ
\end{aligned}$$

where $C = \max\{\lambda \cdot \big[-(u_l^\epsilon + u_r^\epsilon - 2k)\big]\}$. It is also clear that $I \geqslant 0$ as $\epsilon \to 0$. This completes the proof of Theorem 2.1. ■

2.2 Adjoint problem for linear functional

The objective of this thesis is to estimate the error with respect to a linear functional of the form

$$J(u) = \int_{\mathbb{R}} u(x,T) \cdot \zeta_0(x)\, dx, \tag{2.9}$$

where $\zeta_0 \in C_0^\infty(\mathbb{R}, \mathbb{R}^n)$. In other words, given an approximate solution u^ϵ to (2.1), how accurate of an approximation is $J(u^\epsilon)$. This presentation follows unpublished notes from Laforest.

Although we could extend our results to linear functionals depending on space and time, after several modifications of the framework, it is typical to consider this case. When ζ_0 is a positive test function of unit mass, like $\zeta_0(x) \approx C \cdot e^{-(x-x_0)^2}$, then the functional J measures the average value of u in a small neighborhood of x_0. Similarly, when $\zeta_0(x) \approx C \cdot (x - x_0) \cdot e^{-(x-x_0)^2}$, J measures an average of the derivative of u in a neighborhood of x_0. More generally, J could be used to measures the outgoing flux of u at the boundary of a domain.

The solution to equation (2.1) satisfies the *weak form* of the conservation law. This means that it must be a measurable function that satisfies

$$\int_{\mathbb{R}\times[0,T]} u \cdot \psi_t + f(u) \cdot \psi_x \, dxdt + \int_{\mathbb{R}} u_0(x) \cdot \psi(x,0)\, dx = 0 \tag{2.10}$$

for all $\psi \in C_0^\infty(\mathbb{R}\times[0,T),\mathbb{R}^n)$. The regularity of ψ can be relaxed to

$$W^{1,\infty}(\mathbb{R}\times[0,T),\mathbb{R}^n) = \{u \big| \|u\|_{L^\infty} + \|\partial_t u\|_{L^\infty} < \infty\},$$

as long as the condition of compact support remains. Removing the condition of compact support in time (but not in space) and assuming we knew the value of u at time T, we may rewrite the weak form as

$$\int_{\mathbb{R}\times[0,T]} u\cdot\phi_t + f(u)\cdot\phi_x\,dxdt + \int_{\mathbb{R}} u_0(x)\cdot\phi(x,0)\,dx = \int_{\mathbb{R}} u(x,T)\cdot\phi(x,T)\,dx \quad (2.11)$$

for all smooth $\phi \in W_0^{1,\infty}(\mathbb{R}\times[0,T],\mathbb{R}^n)$.

We begin our analysis of the error estimator by defining the weak form of the residual R. Given our coarse approximation u^ϵ, write

$$\begin{aligned}\int_{\mathbb{R}} u^\epsilon(x,T)\cdot\phi(x,T)\,dx = &\int_{\mathbb{R}\times[0,T]} u^\epsilon\cdot\phi_t + f(u^\epsilon)\cdot\phi_x\,dxdt\\ &+\int_{\mathbb{R}} u^\epsilon(x,0)\cdot\phi(x,0)\,dx + R(u^\epsilon,\phi),\end{aligned} \quad (2.12)$$

using the obvious definition

$$\begin{aligned}R(u^\epsilon,\phi) \doteq &-\int_{\mathbb{R}\times[0,T]} u^\epsilon\cdot\phi_t + f(u^\epsilon)\cdot\phi_x\,dxdt\\ &-\int_{\mathbb{R}} u^\epsilon(x,0)\cdot\phi(x,0)\,dx + \int_{\mathbb{R}} u^\epsilon(x,T)\cdot\phi(x,T)\,dx.\end{aligned} \quad (2.13)$$

Suppose that the test function $\zeta_0(\cdot)$ that defines the functional

$$J(u) = \int_{\mathbb{R}} u(x,T)\cdot\zeta_0(x)\,dx, \quad (2.14)$$

can be extended to a function $\zeta \in W^{1,\infty}(\mathbb{R} \times [0,T], \mathbb{R}^n)$. Using identities (2.11) and (2.12), then the error with respect to functional can be written

$$
\begin{aligned}
J(u^\epsilon) - J(u) = &- \int_{\mathbb{R}\times[0,T]} u \cdot \zeta_t + f(u) \cdot \zeta_x \, dxdt - \int_{\mathbb{R}} u(x,0) \cdot \zeta(x,0) \, dx \\
&+ \int_{\mathbb{R}\times[0,T]} u^\epsilon \cdot \zeta_t + f(u^\epsilon) \cdot \zeta_x \, dxdt + \int_{\mathbb{R}} u^\epsilon(x,0) \cdot \zeta(x,0) \, dx \\
&+ R(u^\epsilon, \zeta) \\
= &\int_{\mathbb{R}\times[0,T]} (u^\epsilon - u) \cdot \zeta_t + \big(f(u^\epsilon) - f(u)\big) \cdot \zeta_x \, dxdt \qquad (2.15) \\
&+ \int_{\mathbb{R}} \big(u^\epsilon(x,0) - u(x,0)\big) \cdot \zeta(x,0) \, dx + R(u^\epsilon, \zeta). \qquad (2.16)
\end{aligned}
$$

The two terms in (2.16) are computable using only knowledge of $\zeta(x,t)$, $u^\epsilon(x,t)$ and the initial data $u_0(x)$. This suggests that *if* we can choose the extension ζ in such a way that the term in (2.15) vanishes, i.e.

$$
\begin{aligned}
0 &= \int_{\mathbb{R}\times[0,T]} (u^\epsilon - u) \cdot \zeta_t + \big(f(u^\epsilon) - f(u)\big) \cdot \zeta_x \, dxdt \\
&= \int_{\mathbb{R}\times[0,T]} (u^\epsilon - u) \cdot \Big(\zeta_t + F(u^\epsilon, u)^T \zeta_x\Big) \, dxdt, \qquad (2.17)
\end{aligned}
$$

and *if* ζ is computable entirely from u^ϵ and u_0, then we will have obtained a computable estimate of the error. Clearly though, the definition of ζ will also depend on the entropy solution u. We will show that such a ζ exists in the next chapter. We have to mention here that the well-known nonlinear functional

$$F(u^\epsilon, u) = \int_0^1 Df(su^\epsilon + (1-s)u) \, ds \qquad (2.18)$$

satisfies the identity

$$F(u^\epsilon, u)(u^\epsilon - u) = f(u^\epsilon) - f(u). \qquad (2.19)$$

We also remark that in practice, the expression $F(u^\epsilon, u) = F(u^\epsilon(x,t), u(x,t))$ is a function of (x,t). Keep in mind here, the functional F is actually the shock speed S mentioned in Chapter 1, i.e. $F = S$.

In practice, ζ depends on the exact solution u and therefore cannot be used for a posteriori error estimation. So, in (2.16), we will therefore approximate u by u^η and ζ by $\xi^{\eta,h}$, both variables will be introduced in the next chapter. The result will be

$$\begin{aligned}
J(u^\epsilon) - J(u) &= \int_{\mathbb{R}} \big(u^\epsilon(x,0) - u(x,0)\big)\big(\zeta(x,0) - \xi^{\eta,h}(x,0)\big)\,dx \\
&\quad + \int_{\mathbb{R}} \big(u^\epsilon(x,0) - u(x,0)\big)\xi^{\eta,h}(x,0)\,dx \\
&\quad + R(u^\epsilon, \xi^{\eta,h}) + R(u^\epsilon, \zeta - \xi^{\eta,h}) \\
&= \int_{\mathbb{R}} \big(u^\epsilon(x,0) - u^\eta(x,0)\big)\xi^{\eta,h}(x,0)\,dx \\
&\quad + \int_{\mathbb{R}} \big(u^\eta(x,0) - u(x,0)\big)\xi^{\eta,h}(x,0)\,dx \\
&\quad + \int_{\mathbb{R}} \big(u^\epsilon(x,0) - u(x,0)\big)\big(\zeta(x,0) - \xi^{\eta,h}(x,0)\big)\,dx \\
&\quad + R(u^\epsilon, \xi^{\eta,h}) + R(u^\epsilon, \zeta - \xi^{\eta,h}) \\
&= \int_{\mathbb{R}} \big(u^\epsilon(x,0) - u^\eta(x,0)\big)\xi^{\eta,h}(x,0)\,dx + R(u^\epsilon, \xi^{\eta,h}) \\
&\quad + \int_{\mathbb{R}} \big(u^\eta(x,) - u(x,0)\big)\xi^{\eta,h}(x,0)\,dx \\
&\quad + \int_{\mathbb{R}} \big(u^\epsilon(x,0) - u(x,0)\big)\big(\zeta(x,0) - \xi^{\eta,h}(x,0)\big), dx \\
&\quad + R(u^\epsilon, \zeta - \xi^{\eta,h}) \\
&= \int_{\mathbb{R}} \big(u^\epsilon(x,0) - u^\eta(x,0)\big)\xi^{\eta,h}(x,0)\,dx + R(u^\epsilon, \xi^{\eta,h}) \\
&\quad + \mathcal{O}(\eta) + \mathcal{O}(\epsilon)\cdot\big(\mathcal{O}(\eta) + \mathcal{O}(h)\big) + R\big(u^\epsilon, \big(\mathcal{O}(\eta) + \mathcal{O}(h)\big)\big). \qquad (2.20)
\end{aligned}$$

Above, $\xi^{\eta,h}$ will depend only on u^ϵ, u^η and an additional parameter h but not on u. To obtain an accurate error estimator, it suffices to take η and h small with respect to ϵ, say $\eta = \mathcal{O}(\epsilon^2)$ and $h = \mathcal{O}(\epsilon^2)$, retain only the first two terms in (2.20).

CHAPTER 3

MAIN RESULTS

As mentioned in the previous chapter, this chapter describes the proof of the existence of the function $\xi^{\eta,h}$ satisfying (2.20). In this thesis, we only focus on approximate solutions containing exclusively shock discontinuities. The main results are Theorem 3.1, which contains the proof of the existence of computable piecewise linear approximations of $\xi^{\eta,h}$, as well as Lemmas 3.6 and 3.8 that show that these approximations converge to ζ if a certain stability bound holds. The proof of that stability estimate is still an open problem. Ideas and suggestions on treating the general case will be discussed in the Conclusion.

Recall from Chapter 2, that u is the entropy solution of (2.1), with $n = 1$, and u^ϵ, u^η are piecewise constants front-tracking approximations. Undoubtedly, $0 < \eta \ll \epsilon$ and therefore u^η is a better approximation of u than u^ϵ. A few basic results will be used without comment. Since u^ϵ and $u^\eta \in L^\infty(\mathbb{R})$ for $t \in [0,T]$, f' is continuous and strictly positive, then we have

$$
\begin{aligned}
F(u^\epsilon, u^\eta) &= \int_0^1 f'\big(\lambda u^\epsilon + (1-\lambda)u^\eta\big)\, d\lambda \\
&\leqslant \int_0^1 f'\big(\max\{u^\epsilon, u^\eta\}\big)\, d\lambda \\
&= f'\big(\max\{u^\epsilon, u^\eta\}\big).
\end{aligned}
$$

So, $F(u^\epsilon, u^\eta)$ is bounded. Furthermore, $F(u^\epsilon, u^\eta) = F\big(u^\epsilon(x,t), u^\eta(x,t)\big)$ is also piecewise constant with respect to x and t.

Theorem 3.1 *Suppose that u^ϵ and u^η are front-tracking approximations for initial data containing only shock waves. Suppose that $\{\xi_0^h(\cdot)\}_h$ is a sequence of piecewise linear approximations that converge to $\zeta_0(\cdot)$ in $W^{1,\infty}(\mathbb{R})$ as $h \to 0$. For each positive h, ϵ and η, there exists a piecewise linear strong solution $\xi^{\eta,h} \in W^{1,\infty}(\mathbb{R} \times [0,T])$ of*

$$\begin{aligned} \xi_t^{\eta,h} + F(u^\epsilon, u^\eta)\xi_x^{\eta,h} =& \quad 0 \\ \xi^{\eta,h}(\cdot, T) =& \quad \xi_0^h(\cdot). \end{aligned} \tag{3.1}$$

Conjecture *There exists constants C_1 and C_2, which depend only on f, u_0 and ϵ, such that for any $\eta_1, \eta_2, h_1, h_2,$, the solutions $\xi^{\eta_1,h_1}, \xi^{\eta_2,h_2}$ of* (3.1) *given in Theorem 3.1 satisfy the stability estimate*

$$\begin{aligned} \|\xi^{\eta_1,h_1} - \xi^{\eta_2,h_2}\|_{W^{1,\infty}(\mathbb{R}\times[0,T])} \leq& \quad C_1\|\xi_0^{h_1} - \xi_0^{h_2}\|_{W^{1,\infty}(\mathbb{R})} \\ & +C_2\|u^{\eta_1} - u^{\eta_2}\|_{\mathrm{Lip}([0,T],L^1(\mathbb{R}))}. \end{aligned} \tag{3.2}$$

Despite having not found a proof of the stability estimate, we will show how this estimate allows us to demonstrate the existence of ζ. The derivation of the existence from the stability estimate was described in unpublished notes of Laforest. The original contribution of this thesis is therefore the proof of the adjoint to the problem (3.1).

The first part is the proof of the existence of a piecewise linear $\xi^{\eta,h}$.

3.1 Existence of adjoint $\xi^{\eta,h}$

Definition 3.2 *We say that there is a shock crossing at* (x,t) *if there exists two discontinuities belonging to either* u^ϵ, u^{η_1} *or* u^{η_2}, *not both necessary in the same approximation but both located at* (x,t).

Lemma 3.3 *Suppose two consecutive nodes travel along characteristics* $z_1(t) < z_2(t)$. *Suppose that for some* $\tau \in [0,T]$:

i) there is no shock in either $u^\epsilon(\cdot,T)$ *or* $u^\eta(\cdot,T)$ *between* $z_1(T)$ *and* $z_2(T)$,

ii) the function $\xi^{\eta,h}(\cdot,T)$ *is linear between* $z_1(T)$ *and* $z_2(T)$,

iii) there is no shock crossing in the open region

$$R := \ \{(x,t)|t \in (0,T), z_1(t) < x < z_2(t)\},$$

iv) there is no shock between $z_1(0)$ *and* $z_2(0)$.

Then, the function of $\xi^{\eta,h}(x,0)$ *is linear with respect to* x *for* $x \in [z_1(0), z_2(0)]$.

Proof. Suppose there is no shock crossing in the region R, then the value of ξ_1 at z_1 and the value of ξ_2 at z_2 remain constant along their characteristics, that is the function of $\xi^{\eta,h}(x,0)$ is linear with respect to x for $x \in [z_1(0), z_2(0)]$.

Suppose there are some shocks crossing the region R. Consider the characteristic $X(x,t)$ which is a curve in $\mathbb{R} \times [0,T]$ satisfying

$$X(x,T) = x, \qquad x \in [z_1(T), z_2(T)] \tag{3.3}$$

$$\frac{\partial X}{\partial t}(x,t) = F\big(u^\epsilon\big(X(x,t),t\big), u^\eta\big(X(x,t),t\big)\big), \qquad t \in [0,T]. \tag{3.4}$$

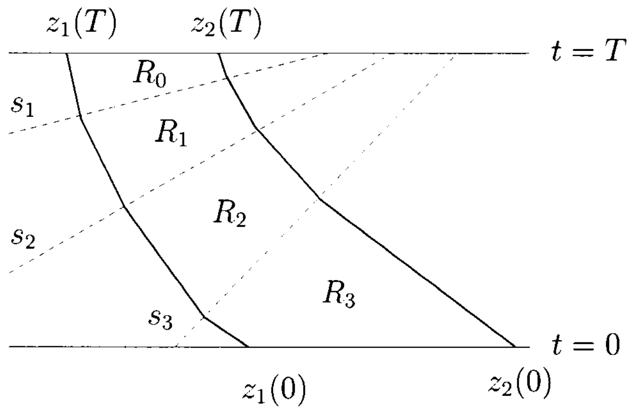

Figure 3.1 A region bounded by characteristics and containing no shock crossings.

If we can show that $X(x,0)$ is linear with respect to x, then $x = x(X,0)$ will be linear with respect to X and, because the value of $\xi^{\eta,h}$ along $X(x,t)$ is constant, we will conclude that

$$\xi^{\eta,h}(X,0) = \xi^{\eta,h}\big(X(x,0),0\big) = \xi^{\eta,h}\big(x(X,0),T\big)$$

is also linear with respect to X. Our goal is therefore to show that $X(x,0)$ is a linear function of x.

As shown in Figure 3.1 when $k = 3$, suppose there are k shocks that cross the region R. Let $s_1(t)$ to $s_k(t)$ be the positions of these shocks inside R. These positions are linear functions of t. Suppose these shocks meet the characteristic z_1 at times

$$t_1^l \geqslant t_2^l \geqslant \cdots \geqslant t_k^l, \tag{3.5}$$

satisfying

$$z_1(t_i^l) = s_i(t_i^l), \qquad i = 1, \ldots, k.$$

The subscript l is used to indicate that this is the time at which the shock crosses the

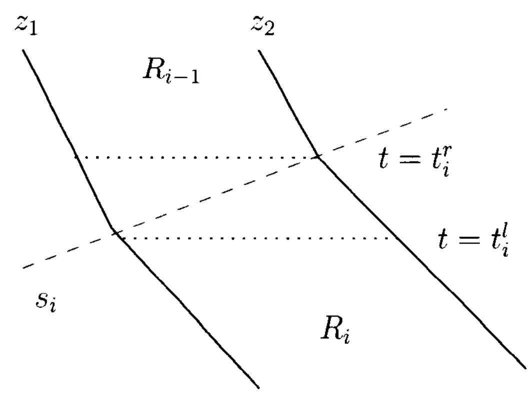

Figure 3.2 A shock crossing two parallel characteristics.

characteristic z_1 on the left, as in Figure 3.1 If t_i^r is the time at which the i^{th} shock meets z_2, that is

$$z_2(t_i^r) = s_i(t_i^r), \qquad i = 1, \ldots, k,$$

then (3.5) and the condition $iii)$ of Lemma 3.3 show that

$$t_1^r \geqslant t_2^r \geqslant \cdots \geqslant t_k^r.$$

Now, we can decompose the whole domain R into three types of trapezoidal regions. The regions will be described by their boundaries.

First, as shown in Figure 3.1, we know the regions R_i for $i = 1, \cdots, k-1$ with

$$\begin{aligned} \partial R_i \doteq \quad & \big(z_1(t), t\big) && \text{for } t \in [t_i^l, t_{i+1}^l], \\ & \big(z_2(t), t\big) && \text{for } t \in [t_i^r, t_{i+1}^r], \\ & \big(s_i(t), t\big) && \text{for } t \in [t_i^l, t_i^r], \\ & \big(s_{i+1}(t), t\big) && \text{for } t \in [t_{i+1}^l, t_{i+1}^r], \end{aligned}$$

then we have the regions abutting the line $t = T$ and $t = \tau$, namely R_0 and R_k, described by

$$\partial R_0 \doteq \begin{cases} (z_1(t), t) & \text{for } t \in [t_1^l, T], \\ (z_2(t), t) & \text{for } t \in [t_1^r, T], \\ (x, T) & \text{for } x \in [z_1(T), z_2(T)], \\ (s_1(t), t) & \text{for } t \in [t_1^l, t_1^r], \end{cases}$$

$$\partial R_k \doteq \begin{cases} (z_1(t), t) & \text{for } t \in [\tau, t_k^l], \\ (z_2(t), t) & \text{for } t \in [\tau, t_k^r], \\ (s_k(t), t) & \text{for } t \in [t_k^l, t_k^r], \\ (x, 0) & \text{for } x \in [z_1(0), z_2(0)]. \end{cases}$$

The approximations u^ϵ and u^η are constant in $R_i, i = 0, 1, \ldots, k$ and therefore $F(u^\epsilon, u^\eta)$ is also constant in each R_i. Suppose that the constant value of $F(u^\epsilon, u^\eta)$ over R_i is denoted F_i. So, from equation (3.4), the characteristic $X(x,t)$ is linear with respect to t within every region R_i.

For the region R_0, the characteristic line $\big(X(x,t), t\big)$ intersects with the shock line $s_1(t)$ at the point $\big(X_1(x), t_1(x)\big)$. The two lines are respectively:

$$\begin{aligned} X(x,t) &= x - F_0(T-t), \\ s_1(t) &= \left(\frac{t - t_1^r}{t_1^l - t_1^r}\right) \cdot s_1(t_1^l) + \left(\frac{t - t_1^l}{t_1^r - t_1^l}\right) \cdot s_1(t_1^r). \end{aligned}$$

Clearly, $X(x,t)$ is a linear function of x and t and $s_1(t)$ is a linear function of t. Therefore, $t_1(x)$ will be a linear function of x after solving $X(x,t) = s_1(t)$. This implies that $X_1(x)$ is linear with respect to x,

$$X_1(x) = X(x, t_1(x)) = x - F_0\big(T - t_1(x)\big).$$

For the region R_i, $i \in [1, k-1]$, assume that the characteristic $X(x,t)$ crosses the i^{th} shock at the point $\big(X_i(x), t_i(x)\big)$ given as a linear function of x. We will show that the next intersecting point $\big(X_{i+1}(x), t_{i+1}(x)\big)$, at which the characteristic $X(x,t)$ meets the $(i+1)^{th}$ shock, is also a linear function of x. Here, the two lines are given by

$$\begin{aligned} X(x,t) &= X_i(x) - F_i(t_i(x) - t), \\ s_{i+1}(t) &= \left(\frac{t - t^r_{i+1}}{t^l_{i+1} - t^r_{i+1}}\right) \cdot s_{i+1}(t^l_{i+1}) + \left(\frac{t - t^l_{i+1}}{t^r_{i+1} - t^l_{i+1}}\right) \cdot s_{i+1}(t^r_{i+1}). \end{aligned}$$

Again, these two lines intersect each other at time $t_{i+1}(x)$ which is clearly a linear function of x because the coefficient of t in both expressions is independent of x. This implies that $X_{i+1}(x)$ is also a linear function of x.

For the last region R_k, the characteristic $X(x,t)$ crosses the time $t = 0$ at the point

$$X(x, 0) = X_k(x) - F_k\big(t_k(x) - 0\big),$$

which is also linear function of x. ■

We now introduce some notation. In the neighborhood of a shock in u^ϵ or u^η, located at

$s_i(t)$, we have the left and right value of the flux

$$F\big(u^\epsilon(x,t),u^\eta(x,t)\big)=\begin{cases} F_l & \text{if } x\leqslant s_i(t),\\ F_r & \text{if } x> s_i(t),\end{cases}$$

where

$$\begin{aligned} F_l &= F(u_l^\epsilon,u_l^\eta)\doteq\int_0^1 f'\big(\lambda u_l^\epsilon+(1-\lambda)u_l^\eta\big)\,d\lambda,\\ F_r &= F(u_r^\epsilon,u_r^\eta)\doteq\int_0^1 f'\big(\lambda u_r^\epsilon+(1-\lambda)u_r^\eta\big)\,d\lambda.\end{aligned}$$

Since we have only shocks and f is convex, it means that $u_l^\epsilon\geqslant u_r^\epsilon$ and $u_l^\eta\geqslant u_r^\eta$. Using the definition of the function $F(u^\epsilon,u^\eta)$, see (2.18), we get $F_l\geqslant F_r$. For each shock, the fluxes satisfy one of the following inequalities:

$$\begin{aligned} F_r\leqslant \dot{s}_i(t)&\leqslant F_l,\\ \dot{s}_i(t)\leqslant F_r\ &\leqslant F_l,\\ F_r\leqslant F_l\ &\leqslant \dot{s}_i(t),\end{aligned}\tag{3.6}$$

where

$$\dot{s}_i(t)=\begin{cases} \frac{f(u_l^\epsilon)-f(u_r^\epsilon)}{u_l^\epsilon-u_r^\epsilon} & \text{if the shock is in } u^\epsilon,\\ \frac{f(u_l^\eta)-f(u_r^\eta)}{u_l^\eta-u_r^\eta} & \text{if the shock is in } u^\eta.\end{cases}$$

Theorem 3.4 *Assume u^ϵ and u^η are front-tracking approximations which have only shocks and*

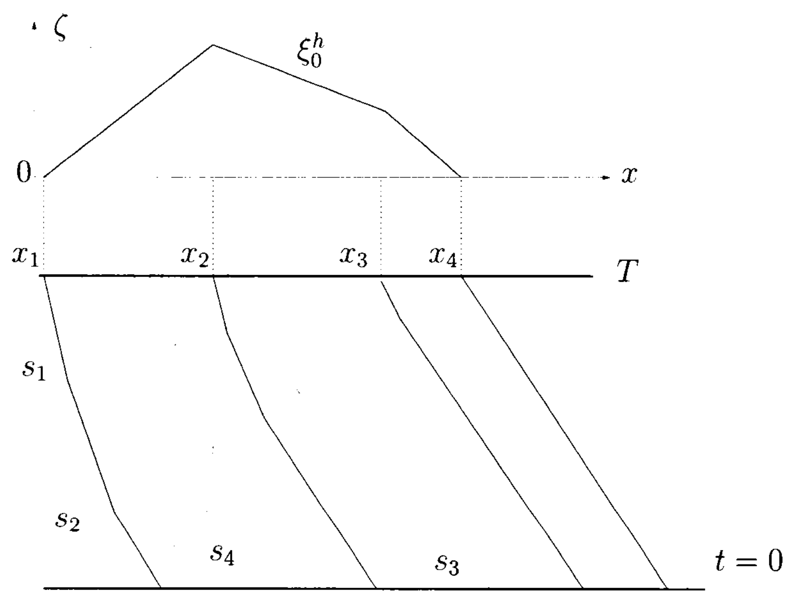

Figure 3.3 Subdivision of $R := \{(x,t)|x \in [x_1, x_N], t \in [0,T]\}$.

$$\lim_{x \to \pm\infty} u^\epsilon(x,t) = \lim_{x \to \pm\infty} u^\eta(x,t).$$

If $\xi_0^{\eta,h}$ is piecewise linear with compact support then there exists a strong solution $\xi^{\eta,h}$ of

$$\xi_t^{\eta,h} + F(u^\epsilon, u^\eta)\xi_x^{\eta,h} = 0 \tag{3.7}$$

$$\xi^{\eta,h}(\cdot, T) = \xi_0^h(\cdot).$$

Moreover, $\xi^{\eta,h}(x,t)$ is also piecewise linear with respect to x for any fixed $t \in [0,T]$.

Proof: The objective of the proof is to reduce the analysis to the situation described in Lemma 3.3. To accomplish this we need to construct a finite number of regions in which no shock crossings occur and in which the initial data at time $t = T$ is piecewise linear. The main difficulty appears at the end when multiple characteristics z_1, z_2 starting at the same point at time $t = T$ may eventually diverge because of shock crossings at an intermediate time.

As shown in Figure 3.1, since $\xi^{\eta,h}(\cdot,T)$ is piecewise linear with compact support, suppose that there are N nodes in $x_i(T)$ such that $\xi^{\eta,h}$ is linear in each set $[x_i, x_{i+1}]$ and vanishes outside of $[x_1, x_N]$. We introduce the abbreviation $\xi_i \doteq \xi^{\eta,h}(x_i^+(T),T)$. Let $x_i(t)$ be the characteristic of (3.7) starting at $x_i(T)$. According to (2.2), we know that there is a finite number of shocks in the space for each $t \in [0,T]$. Let the total number of shocks in u^ϵ and in u^η at time t be $M(t)$. Let $0 < t_K^* < \cdots < t_2^* < t_1^* < T$ be the K times at which a shock crossing occurs in either in u^ϵ or in u^η. This implies that $M(t)$ is constant during each time interval $[t_i^*, t_{i+1}^*)$, where $i = 1, \cdots, K$. Suppose these shocks are located at : $s_1(t), s_2(t), ..., s_{M(0)}(t)$, possibly by letting $s_i(t) = s_j(t)$ after they interact. Let $I(t)$ be the total number of shock crossings that occurred between discontinuities in u^ϵ or u^η during the time interval $[t,T]$.

From time $t = T$ back to time $t = 0$, the values of ξ_1 at x_1 and ξ_N at x_N stay the same along their characteristics as they had at time T. So, for $x < x_1(t)$ or $x > x_N(t)$, $t \in [0,T]$, $\xi^{\eta,h}$ vanishes.

Now, we need to see what happens in $[x_1, x_N]$ for time $t \in [0,T]$.

The function $\xi^{\eta,h}(x,T)$ between any two neighbouring nodes is linear and depends on the values of $\xi^{\eta,h}(x_i,T)$ at the nodes x_i and x_{i+1},

$$\xi^{\eta,h}(x,T) = \xi_i\left(\frac{x - x_{i+1}}{x_i - x_{i+1}}\right) + \xi_{i+1}\left(\frac{x - x_i}{x_{i+1} - x_i}\right). \tag{3.8}$$

Suppose that the $I(0)$ shock crossings occurring during the time interval $[0,T]$ are located at (z_i, t_i), $i = 1, \ldots, I(0)$ and let $P_i(t)$ be the unique characteristic that crosses the point (z_i, t_i). Furthermore, let $S_i(t)$ be the $M(0) + M(T)$ characteristics in $\xi^{\eta,h}$ that originate respectively at the locations of the shocks in u^ϵ or u^η at time $t = 0$ and $t = T$. The characteristics $x_i(t)$, $P_i(t)$ and $S_i(t)$ form a new set $\{y_j(t)\}$ of distinct characteristics, where $j \leqslant N + I(0) + M(0) + M(T)$. Moreover, if a shock crossing occurred at time t^* involving shocks $s_i(t)$ and $s_j(t)$ then one might have $F_r \leqslant \dot{s}_i(t) \leqslant F_l$, in which case, there might be two characteristics going through the crossing point $(s_i(t^*), t^*)$. In that case, we introduce two characteristics y_i and y_{i+1}, one for each side of the shock $S_i(t)$. Now, the region $R = \{(z,t)|t \in [0,T], y_j(t) \leqslant z \leqslant y_{j+1}(t)\}$ satisfies the second hypothesis of Lemma 3.3 because all the nodes of set $\{y_j(T)\}_j$ are in the interval $[x_i(T), x_{i+1}(T)]$. There is no shock crossing in R because the set $\{y_j(t)\}$ include all characteristics through shock crossing. Finally, the last hypothesis of Lemma 3.3 is satisfied for similar reason. In conclusion, if $y_j(T) < y_{j+1}(T)$, then $\xi^{\eta,h}(x,0)$ is linear between $y_j(0)$ and $y_{j+1}(0)$. If $y_j(T) = y_{j+1}(T)$, then it must be because there is a shock $s_i(t)$ such that at time t^*

$$
\begin{aligned}
& y_j(t^*) = s_i(t^*) = y_{j+1}(t^*) \\
\text{and} \qquad & F_r \leqslant s_i(t) \leqslant F_l \qquad \text{for } t < t^*.
\end{aligned}
$$

In that case, we force that $\xi^{\eta,h}$ is constant in R, and therefore again piecewise linear.

Although apparently arbitrary, we impose that $\xi^{\eta,h}$ be constant based on previous results of Bouchut and James (Bouchut and James, 1998) that demonstrated existence and uniqueness of solutions for this problem under the additional (entropy-like) condition that it have smallest total variation. Clearly the constant solution in this reason is the one with the smallest total variation. ■

The following result indicates that in the presence of discontinuities, the characteristics of the adjoint problem separate. This implies that the spatial derivatives of $\xi^{\eta,h}$ must decrease in time, thereby providing some form of stability to these approximations in $W^{1,\infty}$.

Theorem 3.5 *Suppose the characteristic $z(\cdot)$ crosses n shocks (in either u^ϵ, u^{η_1}, or u^{η_2}) during the interval $[t, T]$, each with speed s_i and suppose that the flux $F(u^\epsilon, u^\eta)$ is equal to F_{i-1} and F_i respectively before and after the shock, then*

$$\frac{\partial \xi^{\eta,h}}{\partial x}(z(t), t) = \left(\prod_{i=1}^{n} \frac{F_{i-1} - s_i}{F_i - s_i} \right) \cdot \frac{\partial \xi^{\eta,h}}{\partial x}(z(T), T).$$

The factor is positive and strict less than 1 if $n \geq 1$.

Proof. Suppose from time t, the time interval is Δt during which there is no shock crossing. After Δt, the point x_1, which travels along the characteristic, becomes x_2. Let the crossing point be x_3 where the time line $t + \Delta t$ crosses with the shock s_i and let the distance between x_2 and x_3 be δ', then let x_4 be the point when x_3 travels back along its characteristic at time t. Then, the relations among these four points are

$$\delta = x_4 - x_1 \qquad \delta' = x_3 - x_2$$

$$x_2 = x_1 + F_i \cdot \Delta t \qquad x_3 = x_1 + s_i \cdot \Delta t \qquad x_3 = x_4 + F_{i-1} \cdot \Delta t.$$

So, we have

$$\delta' = (s_i - F_i) \cdot \Delta t = (s_i - F_i) \cdot \frac{\delta}{s_i - F_{i-1}}$$

From time T to t^*, there are n shocks. Then at time t^*, we have

$$\delta'(t^*) = \Big(\prod_{i=1}^{n} \frac{F_{i-1} - s_i}{F_i - s_i}\Big) \cdot \delta(T).$$

Then,

$$\begin{aligned}
\frac{\partial \xi^{\eta,h}}{\partial x}\big(z(t^*), t^*\big) &= \frac{\xi^{\eta,h}\big(z(t^*) + \delta', t^*\big) - \xi^{\eta,h}\big(z(t^*), t^*\big)}{\delta'(t^*)} \\
&= \frac{\xi^{\eta,h}\big(z(T), T\big) - \xi^{\eta,h}\big(z(T), T\big)}{\delta'(t^*)} \\
&= \frac{\delta(T)}{\delta'(t^*)} \cdot \frac{\partial \xi^{\eta,h}}{\partial x}\big(z(T), T\big) \\
&= \frac{\delta(T)}{\Big(\prod_{i=1}^{n} \frac{F_{i-1}-s_i}{F_i - s_i}\Big) \cdot \delta(T)} \cdot \frac{\partial \xi^{\eta,h}}{\partial x}\big(z(T), T\big) \\
&= \Big(\prod_{i=1}^{n} \frac{F_{i-1} - s_i}{F_i - s_i}\Big) \frac{\partial \xi^{\eta,h}}{\partial x}\big(z(T), T\big).
\end{aligned}$$

Since all the shocks crossing the characteristic $z(t)$ satisfy $s_i < F_{i-1} < F_i$ or $F_{i-1} < F_i < s_i$, so, the factor will keep positive and less than 1. We complete the proof. ∎

3.2 Existence of adjoint ζ

In this section, we show how Conjecture (3.2) implies the existence of ζ.

Lemma 3.6 *Suppose that* $\{\xi_0^h(\cdot)\}_h$ *is a sequence of piecewise linear approximations that converge to* $\zeta_0(\cdot)$ *in* $W^{1,\infty}(\mathbb{R})$ *as* $h \to 0$. *If Conjecture* (3.2) *holds, then for each fixed* η *there exists* $\zeta^\eta \in W^{1,\infty}(\mathbb{R} \times [0, T])$ *such that as* $h \to 0$

$$\xi^{\eta,h} \to \zeta^\eta \qquad \textit{in} \qquad W^{1,\infty}(\mathbb{R} \times [0, T]).$$

Using C_2 given in Conjecture (3.2), *ζ^η also satisfies*

$$||\zeta^{\eta_1} - \zeta^{\eta_2}||_{W^{1,\infty}(\mathbb{R}\times[0,T])} \leq C_2||u^{\eta_1} - u^{\eta_2}||_{\mathrm{Lip}([0,T],L^1(\mathbb{R}))}. \tag{3.9}$$

Proof. $W^{1,\infty}(\mathbb{R}\times[0,T])$ is a Banach space. Therefore to demonstrate the existence of a limit, it suffices to show that the sequence $\xi^{\eta,h}$ is Cauchy for fixed η and $h \to 0$. The sequence ζ_0^h converges to ζ_0 in $W^{1,\infty}$ and therefore is Cauchy. This means that for any $\widetilde{\epsilon} > 0$, there exists $H > 0$ such that for any $h_i < H$,

$$||\zeta_0^{h_1} - \zeta_0^{h_2}||_{W^{1,\infty}} \leq \widetilde{\epsilon},$$

and therefore, using (3.2) with $\eta \doteq \eta_1 = \eta_2$

$$||\xi^{\eta,h_1} - \xi^{\eta,h_2}||_{W^{1,\infty}} \leq C_1\widetilde{\epsilon}.$$

This shows that there exists a limit $\zeta^\eta \in W^{1,\infty}(\mathbb{R}\times[0,T])$. Estimate (3.2) shows that

$$\begin{aligned}||\zeta^{\eta_1} - \zeta^{\eta_2}||_{W^{1,\infty}} \leq& ||\zeta^{\eta_1} - \xi^{\eta_1,h}||_{W^{1,\infty}} + ||\xi^{\eta_1,h} - \xi^{\eta_2,h}||_{W^{1,\infty}} \\ &+ ||\xi^{\eta_2,h} - \zeta^{\eta_2}||_{W^{1,\infty}} \\ \leq& C_1||\zeta_0 - \xi_0^h||_{W^{1,\infty}(\mathbb{R})} + C_2||u^{\eta_1} - u^{\eta_2}||_{\mathrm{Lip}([0,T],L^1(\mathbb{R}))} \\ &+ C_1||\xi_0^h - \zeta_0||_{W^{1,\infty}(\mathbb{R})}.\end{aligned}$$

Taking the limit as $h \to 0$, we complete this proof. ■

Before we proceed, we need the following preliminary result.

Lemma 3.7 *If u^ϵ, u^{η_1} and u^{η_2} are front-tracking approximations and $F(u^\epsilon, u^\eta)$ given by* *formula* (2.18)*, then we have*

$$\|F(u^\epsilon, u^{\eta_1}) - F(u^\epsilon, u^{\eta_2})\|_{\mathrm{L}^1(\mathbb{R}\times[0,T])} \leqslant C \cdot \|u^{\eta_1} - u^{\eta_2}\|_{\mathrm{L}^1(\mathbb{R}\times[0,T])}.$$

Proof. Using the mean value theorem, there exists λ_s such that the following holds.

$$\begin{aligned}
&\|F(u^\epsilon, u^{\eta_1}) - F(u^\epsilon, u^{\eta_2})\|_{L^1(\mathbb{R}\times[0,T])} \\
&\quad \leq \int_{\mathbb{R}\times[0,T]} \int_0^1 \Big| f'\big(su^\epsilon - (1-s)u^{\eta_1}\big) - f'\big(su^\epsilon - (1-s)u^{\eta_2}\big)\Big| ds\ dxdt \\
&\quad = \int_{\mathbb{R}\times[0,T]} \int_0^1 \Big| f''\big(su^\epsilon + (1-s)[\lambda_s u^{\eta_1} + (1-\lambda_s)u^{\eta_2}]\big) \cdot \\
&\qquad\qquad \big(su^\epsilon + (1-s)u^{\eta_1} - su^\epsilon - (1-s)u^{\eta_2}\big)\Big| ds\ dxdt \\
&\quad = \int_{\mathbb{R}\times[0,T]} \int_0^1 \Big| f''\big(su^\epsilon + (1-s)[\lambda_s u^{\eta_1} + (1-\lambda_s)u^{\eta_2}]\big) \cdot \\
&\qquad\qquad (1-s)\big(u^{\eta_1} - u^{\eta_2}\big)\Big| ds\ dxdt \\
&\quad \leq \int_{\mathbb{R}\times[0,T]} \max\big\{|f''(u^\epsilon)|, |f''(u^{\eta_1})|, |f''(u^{\eta_2})|\big\} \\
&\qquad\qquad \big|u^{\eta_1}(x,t) - u^{\eta_2}(x,t)\big| \int_0^1 |1-s| ds\ dxdt.
\end{aligned}$$

Since $u^\epsilon, u^{\eta_1}, u^{\eta_2} \in L^\infty(\mathbb{R}\times[0,T])$, then there exists a positive real number M such that

$$\max\{|f''(u^\epsilon)|, |f''(u^{\eta_1})|, |f''(u^{\eta_2})|\} \leq M.$$

Using that $\int_0^1 |1-s| ds = 1/2$, we find that

$$\|F(u^\epsilon, u^{\eta_1}) - F(u^\epsilon, u^{\eta_2})\|_{L^1(\mathbb{R}\times[0,T])} \leq \frac{M}{2} \cdot \|u^{\eta_1} - u^{\eta_2}\|_{L^1(\mathbb{R}\times[0,T])}.$$

We assumed that $f \in C^2(\mathbb{R})$ and that $f'' > 0$. This completes this Lemma. ■

The main result of this section is therefore the existence of a solution satisfying the weak form of the adjoint problem, as first introduced in Section 2.2 with equation (2.17).

Lemma 3.8 *If Conjecture* (3.2) *holds, then the sequence* $\{\zeta^\eta\}_\eta$ *of Lemma 3.6 converges in* $W^{1,\infty}(\mathbb{R}\times[0,T])$ *to a limit* ζ *satisfying* (2.17)*, that is*

$$\int_{\mathbb{R}\times[0,T]} (u^\epsilon - u)\cdot\Big(\zeta_t + F(u^\epsilon,u)^T\zeta_x\Big)\,dxdt = \;0. \tag{3.10}$$

Proof. For all η_1, η_2 we have (3.9). Since $W^{1,\infty}(\mathbb{R}\times[0,T],\mathbb{R}^n)$ is a Banach space, this suffices to show that the sequence is Cauchy and therefore that the limit ζ exists.

To show that (3.10) is satisfied, we begin by showing that the integrand of (3.10) is measurable and in $L^1(\mathbb{R}\times[0,T],\mathbb{R}^n)$. Using the fact that $(u^\epsilon-u)F(u^\epsilon,u) = f(u^\epsilon) - f(u)$, we find

$$\begin{aligned}\Big|\int_{\mathbb{R}\times[0,T]} (u^\epsilon - u)\cdot\big(\zeta_t + F(u^\epsilon,u)\zeta_x\big)dxdt\Big|&\\ \le &\|u^\epsilon-u\|_{L^1(\mathbb{R}\times[0,T])}\cdot\|\zeta_t\|_{L^\infty(\mathbb{R}\times[0,T])}\\ &+\|f(u^\epsilon)-f(u)\|_{L^1(\mathbb{R}\times[0,T])}\cdot\|\zeta_x\|_{L^\infty(\mathbb{R}\times[0,T])}.\end{aligned}$$

Since $f \in C^2$ and f' is increasing, there exists $s \in [0,1]$ such that

$$\begin{aligned}\|f(u^\epsilon)-f(u)\|_{L^1(\mathbb{R}\times[0,T])} &= \int_{\mathbb{R}\times[0,T]} \big|f'(su^\epsilon+(1-s)u)\cdot(u^\epsilon-u)\big|dxdt\\ &\le \sup_{v\in\text{Range}\{u^\epsilon,u\}} \big|f'(v)\big|\cdot\|u^\epsilon-u\|_{L^1(\mathbb{R}\times[0,T])}.\end{aligned}$$

It is therefore essential that $u^\epsilon, u \in L^\infty(\mathbb{R}\times[0,T])$.

The proof of (3.10) now proceeds with the following decomposition over the space $\Omega \doteq \mathbb{R}\times[0,T]$:

$$
\begin{aligned}
&\int_\Omega (u^\epsilon - u)\cdot\left(\zeta_t + F(u^\epsilon,u)^T\zeta_x\right)dxdt \\
&= \int_\Omega (u^\eta - u)^T\left(\zeta_t + F(u^\epsilon,u)^T\zeta_x\right)dxdt && (3.11)\\
&\quad + \int_\Omega (u^\epsilon - u^\eta)^T\left(F(u^\epsilon,u) - F(u^\epsilon,u^\eta)\right)^T\zeta_x\,dxdt && (3.12)\\
&\quad + \int_\Omega (u^\epsilon - u^\eta)^T\left((\zeta-\zeta^\eta)_t + F(u^\epsilon,u^\eta)^T(\zeta-\zeta^\eta)_x\right)dxdt && (3.13)\\
&\quad + \int_\Omega (u^\epsilon - u^\eta)^T\left((\zeta^\eta-\xi^{\eta,h})_t + F(u^\epsilon,u^\eta)^T(\zeta^\eta-\xi^{\eta,h})_x\right)dxdt && (3.14)\\
&\quad + \int_\Omega (u^\epsilon - u^\eta)^T\left(\xi^{\eta,h}_t + F(u^\epsilon,u^\eta)^T\xi^{\eta,h}_x\right)dxdt. && (3.15)
\end{aligned}
$$

We examine each term, one at a time.

The term (3.11) is bounded as

$$
\begin{aligned}
\Big|\int_\Omega (u^\eta - u)^T\left(\zeta_t + F(u^\epsilon,u)^T\zeta_x\right)dxdt\Big|
&\le \|u^\eta - u\|_{L^1(\Omega)}\cdot\|\zeta_t\|_{L^\infty(\Omega)}\\
&\quad + \|u^\eta - u\|_{L^1(\Omega)}\cdot\|\zeta_x\|_{L^\infty(\Omega)}\cdot\|F(u^\epsilon,u)\|_{L^\infty(\Omega)}.
\end{aligned}
$$

This term can be made arbitrarily small by taking η small enough. The term $F(u^\epsilon,u)$ in L^∞ is bounded because F is a continuous function of the bounded values u^ϵ, u.

The second term (3.12) is dealt with in the following manner:

$$\Big|\int_\Omega (u^\epsilon - u^\eta)^T \big(F(u^\epsilon, u) - F(u^\epsilon, u^\eta)\big)^T \zeta_x \, dxdt\Big|$$
$$\leq \|u^\epsilon - u^\eta\|_{L^\infty(\Omega)} \cdot \|\zeta_x\|_{L^\infty(\Omega)} \cdot \|F(u^\epsilon, u) - F(u^\epsilon, u^\eta)\|_{L^1(\Omega)}.$$

Lemma 3.7 shows that the last term in this bound vanishes as $u^\eta \to u$ in $L^1(\mathbb{R}\times[0,T])$.

The term (3.13) is also bounded:

$$\Big|\int_\Omega (u^\epsilon - u^\eta)^T \big((\zeta - \zeta^\eta)_t + F(u^\epsilon, u^\eta)^T (\zeta - \zeta^\eta)_x\big) \, dxdt\Big|$$
$$\leq \|u^\epsilon - u^\eta\|_{L^1(\Omega)} \cdot \Big\|(\zeta - \zeta^\eta)_t\Big\|_{L^\infty(\Omega)}$$
$$+ \|f(u^\epsilon) - f(u^\eta)\|_{L^1(\Omega)} \cdot \Big\|(\zeta - \zeta^\eta)_x\Big\|_{L^\infty(\Omega)}.$$

These terms are bounded because $\zeta^\eta \to \zeta$ in $W^{1,\infty}(\Omega)$.

The term (3.14) is now shown to vanish as $h \to 0$:

$$\Big|\int_\Omega (u^\epsilon - u^\eta)^T \big((\zeta^\eta - \xi^{\eta,h})_t + F(u^\epsilon, u^\eta)^T (\zeta^\eta - \xi^{\eta,h})_x\big) \, dxdt\Big|$$
$$\leq \|u^\epsilon - u^\eta\|_{L^1(\Omega)} \cdot \Big\|(\zeta^\eta - \xi^{\eta,h})_t\Big\|_{L^\infty(\Omega)}$$
$$+ \|f(u^\epsilon) - f(u^\eta)\|_{L^1(\Omega)} \cdot \Big\|(\zeta^\eta - \xi^{\eta,h})_x\Big\|_{L^\infty(\Omega)}.$$

In this case, we proved in Lemma 3.6 that $\xi^{\eta,h} \to \zeta^\eta$ as $\eta \to 0$ inside $W^{1,\infty}(\Omega)$. This shows that their derivatives converge in $L^\infty(\mathbb{R}\times[0,T])$.

By construction of $\xi^{\eta,h}$ based on Theorem 3.1, the differential equation (3.1) is satisfied and therefore the term (3.15) vanishes. In conclusion, for any η and any h, we have the bounds on the terms (3.11)-(3.15). Letting $\eta, h \to 0$ proves (3.10). ■

CONCLUSION

There are a lot of phenomena in the world modeled by conservation laws, such as compressible fluid occurs in airplanes, engines, inertial confinement fusion, and supernova explosion etc. How to simulate accurately for better designs and investigations? Numerical analysis plays an important role here. Unfortunately, some of the simulations are complex and too difficult for current computers. We therefore need efficient ways to compute them.

The adjoint method is used to compute an estimate of the error in a functional of the solution, like blow-up time and downstream pressure. It therefore allows very efficient and adapted approximations to these quantities. For conservation laws, the adjoint method for nonlinear systems of conservation laws, to our knowledge, has not been applied rigourously. The objective of this thesis was to explore the formalism of the adjoint method in a context where the analysis was manageable, namely for front-tracking approximations.

In this thesis, we use the front-tracking method to generate approximate solutions and develop a new scheme to solve the adjoint problem. We have showed the existence of the adjoint when using front-tracking approximations. The existence of the solution to the adjoint problem for entropy solutions holds if a certain stability estimate also holds. The approach used here offers the possibility to be extended to systems of conservation laws. This thesis provides a different, and more restricted, proof of the result of Tadmor (Tadmor, 1991). The main contribution of this thesis is the constructive proof of the adjoint which may be extended into a scheme for numerical purposes. it would be interesting to pursue this avenue of research.

The proof relies on the use of Bouchut and James (Bouchut and James, 1998) uniqueness criterion in order to complete then construction. This criterion plays the obvious role of the analogue of the entropy condition of a conservation law. At the moment, the analogue of such a condition, even for the adjoint of 2×2 systems of conservation laws is unknown.

The partial result Theorem 3.5 indicates that stability in $W^{1,\infty}$ is reasonable. To handle the case with more general initial data, it will be necessary to smooth out the rarefaction shocks in the front-tracking approximations in order to extend the characteristics back in time. For systems of conservation laws, the work of LeFloch and Xin (LeFloch and Xin, 1993) might indicate how to handle the new (and weaker) shock and rarefaction waves created by interactions are intermediate times. On the other hand, the original construction of front-tracking DiPerna for 2×2 systems indicates that the total number of rarefaction waves remains bounded in time. This essential property might allow our explicit construction to be extended to systems.

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